

A remark on kinematic hardening

I. Time-dependent theory of plastic multi-component materials(*)

J. STICKFORTH (BRAUNSCHWEIG)

STARTING from the concept of an intermediate configuration, it is shown that a thermo-mechanical theory with internal state variables q_1, q_2, \dots automatically amounts to a time-dependent theory of plasticity with kinematic hardening (including isotropic hardening) if a linear relation is introduced, which connects the microscopic rates of change \dot{q}_k with the plastic flow tensors stemming from the plastic subelements of the material. This procedure amounts to substituting the variables q_k by internal back-stress tensors $\alpha_{(k)}$ correlated with the plastic subelements. Since the idea of a yield condition has not yet been introduced, it is impossible to fully concretize the laws of workhardening and of plastic flow.

Wychodząc z koncepcji konfiguracji pośredniej wykazano, że termo-mechaniczna teoria z parametrami stanu wewnętrznego q_1, q_2 , automatycznie przechodzi w zależną od czasu teorię plastyczności ze wzmocnieniem kinematycznym (z włączeniem wzmocnienia izotropowego) jeżeli wprowadzi się liniową zależność, która uzależnia mikroskopowe prędkości zmiany \dot{q}_k od tensorów plastycznego płynięcia wywodzących się z plastycznych subelementów materiału. Ta procedura jest równoznaczna z podstawieniem zamiast zmiennych q_k tensorów naprężeń własnych $\alpha_{(k)}$ odnoszących się do subelementów plastycznych. Ze względu na to, że nie wprowadza się pojęcia warunku pastyczności, niemożliwe jest pełne skonkretyzowanie praw wzmocnienia i plastycznego płynięcia.

Исходя из концепции косвенной конфигурации показано, что термомеханическая теория с параметрами внутреннего состояния q_1 и q_2 автоматически переходит в зависящую от времени теорию пластичности (включая изотропное упрочнение), если вводить линейное соотношение, которое ставит в зависимость микроскопические скорости и менений \dot{q}_k от тензоров пластического течения, вытекающих из пластических субэлементов материала. Эта процедура равнозначна подстановке вместо переменных q_k тензоров собственных напряжений $\alpha_{(k)}$, относящихся к пластическим субэлементам. В связи с тем, что не можно полностью уточнить законы упрочнения и пластического течения.

1. Introduction

HARDENING, in the most general sense, is the change of plastic response of materials if they undergo plastic deformations. The main objective of this manuscript is this general hardening, and the adjective kinematic in the title expresses the fact that, from the thermo-mechanical point of view, the interrelation between plastic behaviour and the existence of an internal state necessarily amounts to internal back-stress tensors $\alpha_{(1)}, \dots, \alpha_{(n)}$ similar to the kinematic hardening tensor α of A. YU. ISHLINSKIY [1] and W. PRAGER [2] with n indicating the number of plastic subelements. Therefore, kinematic hardening must be considered as the principal feature of every form of plastic hardening, including so-called isotropic hardening.

(*) The paper has been presented at 3th German-Polish Symposium on Mechanics of Inelastic Solids and Structures, Bad Honnef, September 1984.

The paper presents a thermo-mechanically consistent treatment of plasticity *before* introducing the idea of a yield condition. It does not contain any thought which may not be found in the investigations of many other authors. The decisive step is the combination of thermo-mechanics with an idea of E. KRÖNER [3] concerning the connection of plastic deformation increments with the increments of the internal state variables as well as with the Heyn–Masing–Prandtl idea of a multitude of plastic subelements as used by J. F. BESSELING [4], or Z. MRÓZ [5, 6], for instance. It is only the arrangement of arguments which leads to results not generally recognized in the literature. The time-dependent theory of plasticity thus attained is not yet complete, because neither the law of plastic flow nor the hardening laws determining the tensors $\alpha_{(1)}, \dots, \alpha_{(n)}$ can be specified without introducing the nonthermomechanical concept of a yield condition.

Our treatment of plasticity is founded on three pillars:

(1) On general nonlinear thermodynamics of irreversible processes (TIP) which — though presuming local equilibrium, too — is a theoretically open extension of Onsager's theory;

(2) on Eckart's and Kondo's concept of an intermediate configuration amounting to the multiplicative decomposition of the deformation gradient \mathbf{F} ;

(3) on the concept of an internal state Q which is described by an infinite set of (non-measurable) internal state variables q_1, q_2, \dots with rates \dot{q}_k which are linear functions of the plastic strain rate contributions stemming from the different plastic subelements (1), ..., (n).

As regards the first point, TIP merely represents the most direct nonlinear extrapolation of Onsager's philosophy to look at the dissipation function. Because the structure of this function strongly depends on the decomposition of the velocity gradient, we start with the second point.

2. The concept of an intermediate configuration

If a material undergoes inelastic deformations, these must be conceived as the effect of processes occurring on the microscale, independent of material embedding into Euclidean space on the macro-scale, so that the momentary material configuration resulting from inelastic deformation alone becomes a non-Euclidean one, in general. This so-called *intermediate configuration* is often linked with Kondo's name, though it has been introduced already by C. ECKART [7]. Nevertheless, apparently the first *explicit* analytic formulation of the concept, valid for arbitrary large inelastic and elastic deformations, is due to E. KRÖNER [8] ⁽¹⁾ amounting to a multiplicative decomposition of the so-called deformation gradient \mathbf{F} into an inelastic part $\mathbf{F}_{(i)}$, and into an elastic part $\mathbf{F}_{(e)}$, according to

$$(2.1) \quad \mathbf{F} = \mathbf{F}_{(e)}\mathbf{F}_{(i)}.$$

Taking into account the fundamental lemma of continuum mechanics

$$(2.2) \quad \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{G},$$

⁽¹⁾ Other authors who independently rediscovered the formula (2.1) are J. F. BESSELING [9], and E. H. LEE and D. T. LIU [10].

with \mathbf{G} representing the velocity gradient

$$(2.3) \quad \mathbf{G} = {}^t(\nabla \otimes \mathbf{v})$$

(t indicates transposition, and \mathbf{v} means the velocity field), we deduce from the relation (2.1) an additive decomposition of \mathbf{G} ,

$$(2.4) \quad \begin{aligned} \mathbf{G} &= \mathbf{G}_{(e)} + \mathbf{G}_{(i)}, \\ \mathbf{G}_{(e)} &:= \dot{\mathbf{F}}_{(e)} \mathbf{F}_{(e)}^{-1}, \\ \mathbf{G}_{(i)} &:= \mathbf{F} \mathbf{F}_{(i)}^{-1} \dot{\mathbf{F}}_{(i)} \mathbf{F}^{-1}. \end{aligned}$$

Furthermore, the inelastic velocity gradient $\mathbf{G}_{(i)}$ must be decomposed into a scalar and into a deviatoric part

$$(2.5) \quad \bullet \quad \mathbf{G}_{(i)} = \frac{1}{3} e_{(i)} \mathbf{1} + \Phi,$$

with $\mathbf{1}$ indicating the unit tensor and with

$$(2.6) \quad e_{(i)} = (\ln \det \mathbf{F}_{(i)}),$$

representing the *inelastic expansion* stemming from the formation of *micro-voids*, whereas the deviator Φ means the volume conserving true *plastic flow tensor* effected by yielding of the *plastic subelements* (1), ..., (n) so that

$$(2.7) \quad \Phi = \Phi_{(1)} + \dots + \Phi_{(n)}.$$

The number of plastic subelements should be expected to be very large in general⁽²⁾. Nevertheless, the order of this number is unimportant regarding the following derivations, and for practical purposes even $n = 2$, or even $n = 1$, often represents a useful approximation.

Another additive decomposition of the velocity gradient \mathbf{G} is the well-known Euler–Cauchy–Stokes decomposition

$$(2.8) \quad \begin{aligned} \mathbf{G} &= \mathbf{D} + \mathbf{W}, \\ \mathbf{D} &:= {}^s\mathbf{G}, \quad \mathbf{W} := {}^a\mathbf{G}, \end{aligned}$$

with s and a indicating the symmetric and the skew part, respectively. \mathbf{D} is called the deformation rate tensor, whereas \mathbf{W} is the so-called vorticity tensor, or spin tensor, with the latter name expressing that the rotation tensor Ω , resulting from integrating the differential equation

$$(2.9) \quad \dot{\Omega} = \mathbf{W}\Omega,$$

describes i. material rotation.

It is essential to notice at this instant already that thermodynamical reasoning suggests the plastic flow tensors $\Phi_{(1)}, \dots, \Phi_{(n)}$ to be symmetric so that the total and the elastic spin tensors become identical:

$$(2.10) \quad \mathbf{W}_{(e)} = \mathbf{W}.$$

⁽²⁾ Replacing the neutral word “plastic subelement” by the term “glide system”, our decomposition of \mathbf{G} amounts to a similar decomposition put forward in a paper of E. KRÖNER and C. TEODOSIU [11] concerning viscoplastic deformation of single crystals.

3. The first law of thermomechanics

We now turn to the thermodynamical standpoint, beginning with the first law of thermodynamics which presupposes the mechanical conservation laws of linear momentum, of rotational momentum, and of energy. The first two of these conservation laws amount to Cauchy's first and second law of motion,

$$(3.1) \quad \rho \dot{\mathbf{v}} = \operatorname{div} \mathbf{T} + \rho \mathbf{b},$$

$$(3.2) \quad \mathbf{T} = {}^t \mathbf{T},$$

(ρ — mass-density; \mathbf{b} — body-force vector field per unit mass; \mathbf{T} — Cauchy's stress tensor), whereas the law of energy conservation is equivalent with the existence of a scalar equation of balance of total energy without internal supply of energy,

$$(3.3) \quad \rho \dot{e} = -\operatorname{div} \mathbf{J} + \rho r,$$

with e meaning the specific total energy, \mathbf{J} representing the flux of total energy and r indicating the specific external supply of energy.

Assuming that the body-force \mathbf{b} is derived from a specific potential energy e_{pot} according to

$$(3.4) \quad \mathbf{b} = -\nabla e_{\text{pot}},$$

the first law of thermodynamics states the existence of a specific internal energy u so that

$$(3.5) \quad e = \frac{1}{2} \mathbf{v}^2 + e_{\text{pot}} + u.$$

From Eqs. (3.1)–(3.3) we then obtain the *equation of balance of internal energy*,

$$(3.6) \quad \begin{aligned} \rho \dot{u} &= -\operatorname{div} \mathbf{h} + \mathbf{T} \cdot \mathbf{D} + \rho r, \\ \mathbf{h} &:= \mathbf{J} + \mathbf{T} \mathbf{v}. \end{aligned}$$

The internal supply of internal energy, $\mathbf{T} \cdot \mathbf{D}$, is the so-called stress power, whereas the flux of internal energy, \mathbf{h} , represents the so-called heat flux. (Note that $-\mathbf{T} \mathbf{v}$ means the flux of kinetic energy).

In the first place, the specific internal energy must be a function of the metric change necessary for restoring euclidicity in the actual configuration after deforming into the non-Euclidean intermediate configuration. This metric change is described by the elastic right Cauchy–Green tensor

$$(3.7) \quad \mathbf{C}_{(e)} := {}^t \mathbf{F}_{(e)} \mathbf{F}_{(e)}.$$

Furthermore, the specific internal energy depends on the internal state variables q_1, q_2, \dots as well as on the material *parameters* a_1, \dots, a_N , as, for instance, variable elastic moduli, directions of anisotropy, or concentrations of different phases in alloys, etc. The latter is the phenomenon of *anelasticity*. Whereas the a_i are *measurable* variables, the internal state variables q_1, q_2, \dots are *nonmeasurable* ones, in principle.

NOTE. Attention should be drawn to the thermodynamical distinction between *state* variables, on the one hand, and extents of reaction, on the other hand. As not all authors accept this stringent distinction, a comparison of this paper with the results of other authors is difficult.

4. The fundamental equation of state

In the next step we take into account the first part of the second law of thermodynamics. This postulates the existence of a specific entropy s , if local equilibrium is warranted by the existence of local intensive state variables. Therewith, we arrive at a Gibbsian fundamental equation for the specific internal energy which has the form

$$(4.1) \quad u = u(s, \mathbf{C}_{(e)}; a_i, q_k)$$

so that the absolute temperature ϑ amounts to

$$(4.2) \quad \vartheta = \frac{\partial u}{\partial s}.$$

Equation (4.1) is needed to establish the equation of balance of entropy in the next chapter. After this has been done, the equation of state (4.1) will not be used any more because the experimentalist operates with the temperature only. In order to obtain the final formulae in a practical form, we must introduce the specific free energy

$$(4.3) \quad \psi = u - \vartheta s.$$

The Gibbsian fundamental equation of state for this function follows in the usual way from Eqs. (4.1)–(4.3) in the form

$$(4.4) \quad \psi = \psi(\vartheta, \mathbf{C}_{(e)}; a_i, q_k)$$

so that

$$(4.5) \quad s = -\frac{\partial \psi}{\partial \vartheta},$$

whereas

$$(4.6) \quad \frac{\partial \psi}{\partial \mathbf{C}_{(e)}} = \frac{\partial u}{\partial \mathbf{C}_{(e)}},$$

$$(4.7) \quad \frac{\partial \psi}{\partial a_i} = \frac{\partial u}{\partial a_i},$$

$$(4.8) \quad \frac{\partial \psi}{\partial q_k} = \frac{\partial u}{\partial q_k}.$$

As to the structure of the specific free energy function (4.4), we introduce the *stored-energy hypothesis* assuming that ψ is built up additively from a *specific free elastic strain energy* $\psi_{(e)}$, determining the elastic and anelastic behaviour of the material, independent of its internal state Q , and from a *specific free stored-energy* $\psi_{(q)}$, independent of the state of elastic stress, so that

$$(4.9) \quad \psi = \psi_{(e)}(\vartheta, \mathbf{C}_{(e)}; a_i) + \psi_{(q)}(\vartheta, q).$$

This hypothesis is justified by observing that in cyclic stress-strain curves the slope of the linear elastic curve segments remains unchanged.

5. Derivation of the dissipation function

This chapter is a straight-forward continuation of the preceding one, embracing the bulk of mathematical routine. This amounts to, first, solving Eq. (4.1) with respect to s so that

$$(5.1) \quad s = s(\mathbf{u}, \mathbf{C}_{(e)}; a_i, q_k),$$

and then differentiating this function with respect to time, taking into account Eq. (3.6) describing the balance of internal energy, and recognizing the well-known implicit function theorems, as well as Eqs. (4.6)–(4.8).

We thus obtain the *equation of balance of entropy*,

$$(5.2) \quad \rho \dot{s} = -\operatorname{div}(\mathbf{h}/\vartheta) + \frac{1}{\vartheta}(\delta + \rho r),$$

with δ representing the following rather complex expression:

$$(5.3) \quad \delta = \delta_{\text{con}} + \delta_{\text{loc}},$$

$$\delta_{\text{con}} := \mathbf{h} \cdot \vartheta \nabla \frac{1}{\vartheta},$$

$$\delta_{\text{loc}} := (\mathbf{T} - \boldsymbol{\sigma}) \cdot \mathbf{D} + (\mathbf{G} - \mathbf{G}_{(e)}) \cdot \boldsymbol{\sigma} + \sum_1^N \dot{a}_i \left(-\varrho \frac{\partial \psi}{\partial a_i} \right) + \sum_1^\infty \dot{q}_k \left(-\varrho \frac{\partial \psi}{\partial q_k} \right),$$

where $\boldsymbol{\sigma}$ means the *elastic stress tensor*,

$$(5.4) \quad \boldsymbol{\sigma} := 2\varrho \mathbf{F}_{(e)} \frac{\partial \psi}{\partial \mathbf{C}_{(e)}} {}^t \mathbf{F}_{(e)}.$$

From Eq. (5.2) it is evident that δ/ϑ is the entropy production per unit volume and unit time so that δ is the so-called *dissipation function* consisting of two contributions, a *dissipation function by conduction of heat*, δ_{con} , and a *local dissipation function* δ_{loc} which is caused by local entropy producing processes. The usual way of looking at the latter function lies in recognizing that the terms $\boldsymbol{\sigma} \cdot \mathbf{D}$ and $\mathbf{G} \cdot \boldsymbol{\sigma}$ cancel each other so that

$$(5.5) \quad \delta_{\text{loc}} = \mathbf{T} \cdot \mathbf{D} - \mathbf{G}_{(e)} \cdot \boldsymbol{\sigma} + \sum_1^N \dot{a}_i \left(-\varrho \frac{\partial \psi}{\partial a_i} \right) + \sum_1^\infty \dot{q}_k \left(-\varrho \frac{\partial \psi}{\partial q_k} \right).$$

Though Eqs. (5.3)₂ and (5.5) are mathematically identical, they are quite different from the standpoint of thermodynamics of irreversible processes. Indeed, whereas all left factors in Eq. (5.3)₂ vanish under reversible conditions, this is not true in the case of \mathbf{T} and $\mathbf{G}_{(e)}$ in Eq. (5.5). In this paper we are exclusively dealing with the representation (5.3)₂ of the local dissipation function.

Evidently, the left factor of the first term in Eq. (5.3)₂ has the same structure as the definition (3.6a) of the heat flux \mathbf{h} so that the stress difference

$$(5.6) \quad \boldsymbol{\zeta} := \mathbf{T} - \boldsymbol{\sigma}$$

represents the negative irreversible flux of linear momentum which is called the *stress tensor of internal friction*. As to the second term in Eq. (5.3)₂, the left factor $\mathbf{G} - \mathbf{G}_{(e)}$

means the inelastic velocity gradient $\mathbf{G}_{(i)}$ which must be interpreted as *spatial slippage*, and which splits up according to Eqs. (2.5) and (2.6). Decomposing the elastic stress tensor $\boldsymbol{\sigma}$ into its scalar and deviatoric parts,

$$(5.7) \quad \boldsymbol{\sigma} = -p\mathbf{1} + \boldsymbol{\tau},$$

with p meaning the *thermodynamic pressure* and $\boldsymbol{\tau}$ indicating the *elastic shear-stress tensor*, we thus arrive at the following bilinear representation of the local dissipation function:

$$(5.8) \quad \delta_{loc} = \boldsymbol{\zeta} \cdot D + e_{(i)}(-p) + \boldsymbol{\Phi}_{(1)} \cdot \boldsymbol{\tau} + \dots + \boldsymbol{\Phi}_{(n)} \cdot \boldsymbol{\tau} + \sum_1^N \dot{a}_i \left(-\varrho \frac{\partial \psi(e_i)}{\partial a_i} \right) + \sum_1^\infty \dot{q}_k \left(-\varrho \frac{\partial \psi(q)}{\partial q_k} \right),$$

if, furthermore, the stored-energy hypothesis Eq. (4.9) is taken into account.

From this stage on we shall omit internal friction, inelastic expansion, and anelasticity, as expressed by the fluxes $\boldsymbol{\zeta}$, $e_{(i)}$, and \dot{a}_i , respectively. In the following discussion we thus confine ourselves to the reduced local dissipation function describing only pure plastic flow:

$$(5.9) \quad \delta_p = \boldsymbol{\Phi}_{(1)} \cdot \boldsymbol{\tau} + \dots + \boldsymbol{\Phi}_{(n)} \cdot \boldsymbol{\tau} + \sum_1^\infty \dot{q}_k \left(-\varrho \frac{\partial \psi(q)}{\partial q_k} \right).$$

We shall call this the *dissipation function of plastic flow*.

6. Thermodynamics of irreversible processes and its application to plastic flow

Now we turn to the second part of the second law of thermodynamics which states that δ_{loc} is positive,

$$(6.1) \quad \delta_{loc} > 0,$$

if any irreversible process occurs, i.e. if any of the “fluxes” (left-factors) does not vanish.

Just like in Onsager’s linear theory we have a dissipation function of the bilinear type,

$$(6.2) \quad \delta_{loc} = \sum_1^m J_k X_k,$$

with irreversible fluxes J_k which are caused by thermodynamic forces X_k . Presuming *independent fluxes* J_1, \dots, J_m , the main idea of thermodynamics of irreversible processes is the assumption that the fluxes are suitable functions of the forces,

$$(6.3) \quad J_k = J_k(X_1, \dots, X_m; \vartheta, \dots), \quad k = 1, \dots, m,$$

with the adjective “suitable” meaning that δ_{loc} is positive for all imaginable values of the forces X_1, \dots, X_m .

Especially, in linear thermodynamics this assumption amounts to

$$(6.4) \quad J_k = \sum_{l=1}^m L_{kl} X_l$$

with the symmetric part of the L_{kl} -matrix being positive,

$$(6.4) \quad {}^s(L_{kl}) = \text{positive matrix.}$$

However, it is a well-known fact that this general linear thermodynamics of irreversible processes still is an open theory so that further restrictions regarding the L_{kl} are needed, as is the case with the so-called *Onsager-Casimir reciprocity relations* which are derived from the principle of microscopic reversibility. Likewise, it is a well-known fact that there is no general nonlinear extension of Onsager's theory existing (cf. B. H. LAVENDA [12], for instance).

Accordingly, regarding the dissipation function of plastic flow Eq. (5.9), two items are of primary importance:

- (1) The fluxes $\Phi_{(1)}, \dots, \Phi_{(n)}$ and $\dot{q}_1, \dot{q}_2, \dots$ are not independent.
- (2) Plasticity is a nonlinear phenomenon.

As to the first item we assume a linear relation connecting the internal state rates \dot{q}_k with the plastic flow tensors $\Phi_{(1)}, \dots, \Phi_{(n)}$, as well as with the inelastic expansion $e_{(i)}$,

$$(6.5) \quad \dot{q}_k = \sum_{j=1}^n \mathbf{g}_{(j)k} \cdot \Phi_{(j)} + g_k e_{(i)} + \frac{d_{\text{rec}} q_k}{dt},$$

with the last term representing *recovery*, thus generalizing an idea of E. KRÖNER (cf. [3], p. 370). Inserting Eq. (6.5) into Eq. (5.9), and neglecting both, the contribution from $e_{(i)}$ as well as recovery, we obtain the following expression for the dissipation function of plastic flow:

$$(6.6) \quad \delta_p = \Phi_{(1)} \cdot (\boldsymbol{\tau} - \boldsymbol{\alpha}_{(1)}) + \dots + \Phi_{(n)} \cdot (\boldsymbol{\tau} - \boldsymbol{\alpha}_{(n)}),$$

where the tensors

$$(6.7) \quad \boldsymbol{\alpha}_{(j)} := \sum_1^{\infty} \mathbf{g}_{(j)k} \varrho \frac{\partial \psi(q)}{\partial q_k}, \quad j = 1, \dots, n,$$

indicate the *internal back-stresses* which are produced if the plastic subelements are yielded. According to their physical meaning, we expect the coefficients $\mathbf{g}_{(j)k}$ to depend primarily on the internal state, apart from temperature,

$$(6.8) \quad \mathbf{g}_{(j)k} = \mathbf{g}_{(j)k}(\vartheta, q),$$

so that the same is true with the tensors $\boldsymbol{\alpha}_{(j)}$,

$$(6.9) \quad \boldsymbol{\alpha}_{(j)} = \boldsymbol{\alpha}_{(j)}(\vartheta, q).$$

NOTE. From the standpoint of the thermodynamical theory of chemical reactions (cf. J. KESTIN [13], for instance), the plastic subelements (1), ..., (n) must be looked at as n independent chemical reactions with the reactant mole numbers q_1, q_2, \dots . The plastic flow tensors $\Phi_{(1)}, \dots, \Phi_{(n)}$ are representing the respective *rates of reaction* with the $\mathbf{g}_{(j)k}$ meaning the *stoichiometric coefficients*, whereas the $\boldsymbol{\alpha}_{(j)}$ must be interpreted as the *affinities*.

Only deviatoric tensors $\boldsymbol{\alpha}_{(j)}$ must be considered, and because the $\boldsymbol{\alpha}_{(j)}$ appear on the same footing as the elastic shear stress $\boldsymbol{\tau}$, we assume the $\boldsymbol{\alpha}_{(j)}$ to be symmetric, too, so that

$$(6.10) \quad \begin{aligned} \boldsymbol{\alpha}_{(j)} &= {}^t\boldsymbol{\alpha}_{(j)}, & j &= 1, \dots, n, \\ \text{tr} \boldsymbol{\alpha}_{(j)} &= 0, & j &= 1, \dots, n. \end{aligned}$$

In order to obtain the rate equations for the back-stress tensors $\alpha_{(j)}$, we should differentiate Eq. (6.9). However, we must take account of the fact that the $\alpha_{(j)}$ are *material tensors*, and that the same is true with the internal state variables q_k , or, at least a subset of them. As it is well known, rates of material tensors can only be formulated in the corotating system of the moving particle. Therefore, neglecting change of temperature, as well as, for simplicity, inelastic expansion and recovery (the latter will be introduced later), we derive from Eq. (6.5) an expression for the Jaumann derivative only, namely,

$$(7.1) \quad \overset{\nabla}{\alpha}_{(j)} = \sum_{i=1}^n \kappa_{(ji)} \cdot \Phi_{(i)}$$

with material fourth-order tensors $\kappa_{(ji)}$, which are called the *tensors of hardening moduli*, whereas Eq. (7.1) represents the *law of workhardening*. Clearly, the tensors $\kappa_{(ji)}$ indicate the central point of interweaving macroscopic theory with microscopic kinetic models. As to the Jaumann derivative of an arbitrary tensor \mathbf{A} , this is defined by

$$(7.2) \quad \mathbf{A}^{\cdot} = \mathbf{A}^{\nabla} + \mathbf{W}\mathbf{A} - \mathbf{A}\mathbf{W}$$

with \mathbf{W} representing the vorticity tensor Eq. (2.8)₂.

Next, we consider the rates of change of the *effective stress-intensity squares* $\|\tau_{(j)}\|^2$ on account of hardening alone,

$$(7.3) \quad \begin{aligned} (\|\tau_{(j)}\|^2)^{\cdot}|_{\text{hard}} &= -2\tau_{(j)} \cdot \dot{\alpha}_{(j)}, \\ &= -2\tau_{(j)} \cdot \overset{\nabla}{\alpha}_{(j)}. \end{aligned}$$

According to experimental experience, we demand that these rates of change must be negative

$$(7.4) \quad (\|\tau_{(j)}\|^2)^{\cdot}|_{\text{hard}} < 0.$$

Thus, affirming the *hypothesis of independent plastic subelements*, this demand suggests to assume that

$$(7.5) \quad \kappa_{(ji)} = \delta_{ji} \kappa_{(j)}$$

with δ_{ji} indicating Kronecker's symbol so that

$$(7.6) \quad (\|\tau_{(j)}\|^2)^{\cdot}|_{\text{hard}} = -2\tau_{(j)} \cdot (\kappa_{(j)} \cdot \Phi_{(j)}),$$

because, on account of the thermodynamic condition Eq. (6.15), Eq. (7.4) is warranted then by simply demanding that

$$(7.7) \quad \tau_{(j)} \cdot (\kappa_{(j)} \cdot \Phi_{(j)}) = \kappa_{(j)} \tau_{(j)} \cdot \Phi_{(j)} \equiv \kappa_{(j)} \delta_{(j)}$$

with $\kappa_{(j)}$ representing a positive scalar

$$\kappa_{(j)} > 0.$$

This is effected most directly by assuming the projection

$$(7.8) \quad \kappa_{(j)} = \kappa'_{(j)} \left(\mathbf{S} - \frac{1}{3} \mathbf{1} \otimes \mathbf{1} \right) + \kappa''_{(j)} \frac{\tau_{(j)} \otimes \tau_{(j)}}{\tau_{(j)} \cdot \tau_{(j)}}$$

with \mathbf{S} representing the fourth order unit tensor on the space of symmetric second-order tensors so that

$$(7.9) \quad \kappa_{(j)} = \kappa'_{(j)} + \kappa''_{(j)}.$$

If $\kappa''_{(j)}$ vanishes, we obtain a *Prager hardening law* [2], whereas vanishing $\kappa'_{(j)}$ amounts to a *Ziegler hardening law* [14]. Further decisions cannot be made at this stage.

Finally, we may ask how to modify the workhardening law thus arrived at, if additionally, recovery must be taken into account. Most naturally, this can be done by assuming

$$(7.10) \quad \overset{\nabla}{\alpha}_{(j)} = \kappa_{(j)} \cdot \Phi_{(j)} - \gamma_{(j)} e^{-\frac{Q_{(j)}}{R\theta}} \alpha_{(j)}, \quad j = 1, \dots, n$$

with positive *coefficients of recovery*

$$(7.10') \quad \gamma_{(j)} > 0,$$

and with $Q_{(j)}$ representing the *activation energy of recovery in the j -th plastic subelement*.

8. Retrospect and outlook

In the preceding chapters a time-dependent theory has been developed for plastic materials consisting of a multitude of plastic subelements. This theory is determined by a well-defined complete set of evolution equations which we are going to compile now.

In the first instance we possess the two classical evolution equations of continuum mechanics, the trivial equation

$$(8.1) \quad \dot{\mathbf{r}} = \mathbf{v},$$

and Cauchy's first law of motion, Eq. (3.1),

$$(8.2) \quad \rho \dot{\mathbf{v}} = \text{div} \mathbf{T} + \rho \mathbf{b},$$

with the symmetric Cauchy stress-tensor \mathbf{T} being given as the sum of two symmetric material tensor functions,

$$(8.3) \quad \mathbf{T} = \boldsymbol{\sigma} + \boldsymbol{\zeta},$$

where

$$(8.4) \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{F}_{(e)}; \vartheta)$$

represents the elastic stress tensor according to Eq. (5.4), whereas

$$(8.5) \quad \boldsymbol{\zeta} = \boldsymbol{\zeta}(\mathbf{D}; \vartheta),$$

with \mathbf{D} meaning the deformation rate tensor

$$(8.6) \quad \mathbf{D} = {}^s(\nabla \otimes \mathbf{v}),$$

indicates the stress tensor of internal friction which, however, has been omitted in this paper.

In the second place the decomposition of the deformation gradient \mathbf{F} according to Eq. (2.1) affords a further tensorial evolution equation, either for $\mathbf{F}_{(e)}$, or for $\mathbf{F}_{(t)}$. Because of the law of elasticity Eq. (8.4), we opt for $\mathbf{F}_{(e)}$. From Eqs. (2.4) and (2.4)₁ we obtain

$$(8.7) \quad \dot{\mathbf{F}}_{(e)} = ({}^t(\nabla \otimes \mathbf{v}) - \mathbf{G}_{(t)}) \mathbf{F}_{(e)},$$

or, in the case of neglecting inelastic expansion,

$$(8.8) \quad \dot{\mathbf{F}}_{(e)} = ({}^t(\nabla \otimes \mathbf{v}) - \Phi) \mathbf{F}_{(e)},$$

with

$$(8.9) \quad \Phi = \Phi(\boldsymbol{\tau}, \boldsymbol{\alpha}_{(1)}, \dots, \boldsymbol{\alpha}_{(n)}, \vartheta)$$

representing a deviatoric symmetric tensor-valued material function of the elastic shear stress $\boldsymbol{\tau}$, and, apart from temperature, of the internal back-stresses $\boldsymbol{\alpha}_{(1)}, \dots, \boldsymbol{\alpha}_{(n)}$ describing the plastic subelements (1), ..., (n).

NOTE. Usually other authors opt for $\mathbf{F}_{(i)}$ instead of $\mathbf{F}_{(e)}$, thus drawing from Eq. (2.4)₃ an evolution equation for $\mathbf{F}_{(i)}$ which reads

$$(8.10) \quad \dot{\mathbf{F}}_{(i)} = \mathbf{F}_{(i)} \mathbf{F}^{-1} \mathbf{G}_{(i)} \mathbf{F}.$$

However, a procedure of integration based upon this equation is much more troublesome, first, because Eq. (2.2) must be used as a further differential equation then, and secondly, because the law of elasticity Eq. (8.4) becomes very complicated if we substitute

$$(8.11) \quad \mathbf{F}_{(e)} = \mathbf{F} \mathbf{F}_{(i)}^{-1}.$$

A direct confirmation of this statement may be obtained by comparing the author's two papers [15] and [16].

In the third place, we have derived an evolution equation for the specific entropy, namely the equation of balance of entropy Eq. (5.2). However, instead of an equation for the specific entropy s , we need an evolution equation for the temperature ϑ , which is obtained from Eq. (5.2) by differentiating Eq. (4.5) with respect to time. We are thus lead to the rather complex *differential equation of heat conduction*

$$(8.12) \quad \rho c_v \dot{\vartheta} = -\operatorname{div} \mathbf{h} + \delta_{100} + \rho r + \vartheta \frac{\partial \sigma}{\partial \vartheta} \cdot (\mathbf{D} - \mathbf{G}_{(i)}) \\ + \rho \vartheta \sum_1^N \frac{\partial^2 \psi_{(e)}}{\partial \vartheta \partial a_i} \dot{a}_i + \rho \vartheta \sum_1^\infty \frac{\partial^2 \psi_{(q)}}{\partial \vartheta \partial q_k} \dot{q}_k,$$

where

$$(8.13) \quad c_v := -\vartheta \frac{\partial^2 \psi}{\partial \vartheta^2}$$

means the specific heat capacity at constant $\mathbf{F}_{(e)}$, a_i , and q_k . The anelastic fluxes \dot{a}_i must be looked at as given constitutive functions, whereas the internal state rates \dot{q}_k must be substituted according to Eq. (6.5). Further discussing this heat equation, however, surpasses the present concern.

Finally, beside the evolution equations for position \mathbf{r} , for velocity \mathbf{v} , for the elastic deformation gradient $\mathbf{F}_{(e)}$, for temperature ϑ , and eventually for the anelastic material parameters a_i , we have arrived at evolution equations for the internal back-stress tensors $\boldsymbol{\alpha}_{(1)}, \dots, \boldsymbol{\alpha}_{(n)}$, as expressed by Eqs. (7.8) and (7.10) which represent the *law of workhardening including recovery*.

We have thus attained a complete set of evolution equations, only leaving over the question how to specialize the law of plastic flow Eq. (6.14) in accordance with physical

reality, and how to specify the tensor of hardening moduli, as represented by Eq. (7.8). To answer these questions we must introduce the concept of a yield condition, as will be shown in a forthcoming paper.

References

1. A. YU. ISHLINSKIJ, *General theory of plasticity with linear hardening* [in Russian], *Ukrainskij Mat. Zhurnal*, **6**, 314–325, 1954.
2. W. PRAGER, *The theory of plasticity: A survey of recent achievements*, *Proc. Instn. Mech. Engrs.*, **169**, 41–57, 1955.
3. E. KRÖNER, *Plastizität und Versetzungen*, In: A. SOMMERFELD [ed.], *Vorlesungen über Theoretische Physik. Bd. II. Mechanik der deformierbaren Medien*. 5th edition, chapter IX. Akademische Verlagsgesellschaft, Leipzig 1964.
4. J. F. BESSELING, *A theory of elastic, plastic, and creep deformations of an initially isotropic material showing anisotropic strain-hardening, creep recovery, and secondary creep*, *J. Appl. Mech. Trans. ASME*, **25**, 529–536, 1958.
5. Z. MRÓZ, *On the description of anisotropic workhardening*, *J. Mech. Phys. Solid*, **15**, 163–175, 1967.
6. Z. MRÓZ, *An attempt to describe the behavior of metals under cyclic loads using a more general workhardening model*, *Acta Mech.*, **7**, 199–212, 1969.
7. C. ECKART, *The thermodynamics of irreversible processes. IV. The theory of elasticity and anelasticity*, *Phys. Rev.*, **73**, 2, 373–382, 1948.
8. E. KRÖNER, *Allgemeine Theorie der Versetzungen und Eigenspannungen*, *Arch. Rat Mech. Anal.*, **4**, 273–334, 1960.
9. J. F. BESSELING, *A thermodynamic approach to rheology*, In: H. PARKUS, L. I. SEDOV (eds.), *Irreversible Aspects of Continuum Mechanics. Transfer of Physical Characteristics in Moving Fluids*, IUTAM Symposia, Vienna, 1966. Springer-Verlag, Wien 1968.
10. E. H. LEE, D. T. LIU, *Finite strain elastic-plastic theory*. In: Cf. [9].
11. E. KRÖNER, C. TEODOSIU, *Lattice defect approach to plasticity and viscoplasticity*, In: A. SAWCZUK [ed.], *Problems of Plasticity*, International Symposium on Foundations of Plasticity, Warsaw, 1972. Noordhoff International Publishing, Leyden 1974.
12. B. H. LAVENDA, *Thermodynamics of irreversible processes*, The MacMillan Press, London 1978.
13. J. KESTIN, *A course in thermodynamics*, Vol. 2, McGraw-Hill, New York 1979.
14. H. ZIEGLER, *A modification of Prager's hardening rule*, *Quart. Appl. Math.*, **17**, 55–65, 1959.
15. J. STICKFORTH, *New viscoelastic fluid theory based upon the concept of an intermediate state*, *Int. J. Engng Sci.*, **19**, 1775–1788, 1981.
16. J. STICKFORTH, *The rational mechanics and thermodynamics of polymeric fluids based upon the concept of a variable relaxed state*, *Rheol. Acta* [forthcoming].

MECHANIKZENTRUM, TECHNISCHE UNIVERSITÄT BRAUNSCHWEIG, BRD.

Received March 13, 1985.