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# On the initial layer and the existence theorem for the nonlinear Boltzmann equation; differentiability of the solution of the corresponding system of linear equations

M. LACHOWICZ (WARSZAWA)

A LINEAR system of integro-differential equations, which appears in the asymptotic analysis of the nonlinear Boltzmann equation in [9], is considered and the local existence and differentiability of its solution is obtained. Moreover, the estimates, which are necessary in [9], are shown.

W pracy bada się pewien liniowy układ równań całkowo-różniczkowych, pojawiający się w asymptotycznej analizie nieliniowego równania Boltzmanna, przeprowadzonej w [9]. Dowodzi się lokalnego istnienia i różniczkowalności jego rozwiązań oraz prawdziwości oszacowań niezbędnych do analizy w [9].

В работе исследуется некоторая нелинейная система интегро-дифференциальных уравнений, появляющаяся в асимптотическом анализе нелинейного уравнения Больцмана, проведенном в [9]. Доказывается локального существования и дифференцируемости его решений, а также истинности оценок необходимых для анализа в [9].

## 1. Preliminaries

IN [9] WE HAVE considered the truncated Hilbert expansion including the initial layer terms and replaced the singular perturbed Boltzmann equation by a weakly nonlinear system of equations. However, specific smoothness properties of the solution of a corresponding system of linear equations (named CSLE) were required. In the present paper we prove these properties to be true and show the estimates which are necessary in [9]. Similarly as in [9], real-valued functions defined in $I \times \Omega \times R^3$ are considered, where $I$ is an interval and $\Omega$ is a $d$-dimensional torus (in this paper, without losing the generality, we assume $d = 3$). Then $t \in I$ is the time, $x \in \Omega$ is the position of a particle in a rectangular domain $[0, p_1] \times [0, p_2] \times [0, p_3]$ and $\xi \in R^3$ is the velocity. $\Omega$ may be treated as a torus because we assume that all functions encountered in our analysis are periodic with the fundamental domain $[0, p_1] \times [0, p_2] \times [0, p_3]$ (for simplicity we assume $p_1 = p_2 = p_3 = 1$).

Let us denote by $J$ the symmetric bilinear collision operator

$$J(q, r) = J_+(q, r) - J_-(q, r),$$

where

$$J_+(q, r)(\xi) = \frac{1}{2} \int_{R^3} \int_{S^2} k(\xi, \xi_*, \mathfrak{n}) \big(q(\xi') r(\xi'_*) + q(\xi'_*) r(\xi')\big) d\mathfrak{n}\, d\xi_*,$$

$$J_-(q, r) = \frac{1}{2} (q \cdot \nu_r + r \cdot \nu_q) \quad \text{and} \quad \nu_q(\xi) = \int_{R^3} \int_{S^2} k(\xi, \xi_*, \mathfrak{n}) q(\xi_*) d\mathfrak{n}\, d\xi_*$$

The notation is conventional ([2, 3, 5]), $\xi$ and $\xi_*$ denote velocities of the colliding particles before the collision, and

$$\xi' = \xi + \mathfrak{n}(\mathfrak{n}(\xi - \xi_*)), \qquad \xi'_* = \xi_* - \mathfrak{n}(\mathfrak{n}(\xi - \xi_*))$$

after the collision; here $\mathfrak{n} \in S^2$.

Assume that the collision kernel $k$ corresponds to the Grad's cutoff hard potentials.

CARLEMAN [2] proved that the unique solution of

$$J(f_0, f_0) = 0 \tag{1.1}$$

is the function

$$f_0(\xi) = \varrho(2\pi T)^{-\frac{3}{2}} \exp\left(-\frac{|\xi - u|^2}{2T}\right),$$

where $\varrho$, $T$ and $u$ are called the fluid-dynamical parameters of $f_0$ and may depend on $t$ and $x$. If $\varrho$, $T$ and $u$ are constant in $t$ and $x$, $f_0$ is called a global Maxwellian, while in other cases it is called a local Maxwellian. To distinguish them we denote a global Maxwellian by $M = M(\xi)$.

The fluid-dynamical parameters of $f_0$ are assumed to have the following property:

(A1.1) The functions $\varrho$, $u$ and $T$ are smooth enough in $[0, t_0] \times \Omega$ and satisfy the conditions

$$0 < c_\varrho \leqslant \inf_{\substack{t \in [0, t_0] \\ x \in \Omega}} \varrho(t, x),$$

$$0 < c_T \leqslant \inf_{\substack{t \in [0, t_0] \\ x \in \Omega}} T(t, x),$$

where $c_\varrho$, $c_T$ are constants.

A simple consequence of (A1.1) is the existence of positive constants $c^-$ and $c^+$ such that

$$c^- M_- \leqslant w_\alpha f_0 \leqslant c^+ M_+ \qquad \forall t \in [0, t_0], \quad x \in \Omega, \quad \alpha \in R^1, \tag{1.2}$$

where, throughout the paper, $w_\alpha$ denotes the following function

$$w_\alpha(\xi) = (1 + |\xi|^2)^{\alpha/2}. \tag{1.3}$$

Let us introduce the following function spaces. As usual, the space of (real-valued) functions the second power of which is integrable in $R^3$ is denoted by $L_2(R^3)$. The norm in $L_2(R^3)$ is denoted by $\|\cdot; L_2(R^3)\|$. $B^\beta_\infty$ denotes the space of continuous real-valued function on $R^3$ with the weighted norm $\|q; B^\beta_\infty\| = \sup_{\xi \in R^3} |w_\beta| q$. $H^k_2(\Omega)$ is a usual Sobolev space equipped with the norm

$$\|q; H^k_2(\Omega)\| = \left( \sum_{0 \leqslant |\gamma| \leqslant k} \int_\Omega \left( \frac{\partial^{|\gamma|} q}{\partial x^\gamma} \right)^2 dx \right)^{1/2},$$

where

$$\frac{\partial^{|\gamma|}}{\partial x^\gamma} = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \partial x_2^{\gamma_2} \partial x_3^{\gamma_3}}, \qquad |\gamma| = \gamma_1 + \gamma_2 + \gamma_3.$$

$C^k(\Omega)$ is a space consisting of functions which, together with all their derivatives of orders $|\gamma| \leqslant k$, are continuous and equipped with the norm

$$||q; C^k(\Omega)|| = \sup_{\substack{0 \leqslant |\gamma| \leqslant k \\ x \in \Omega}} \left| \frac{\partial^{|\gamma|} q}{\partial x^\gamma} \right|.$$

Finally, introduce the spaces $X^{\beta, k}_{\infty, 2}$, $X^{\beta, k}_{\infty, \infty}$ and $X^{0, k}_{2, 2}$ consisting of real-valued functions on $\Omega \times R^3$ and equipped with the norms

$$||q; X^{\beta, k}_{\infty, 2}|| = ||(||q; H^k_2(\Omega)||); B^\beta_\infty||,$$
$$||q; X^{\beta, k}_{\infty, \infty}|| = ||(||q; C^k(\Omega)||); B^\beta_\infty||,$$
$$||q; X^{0, k}_{2, 2}|| = ||(||q; H^k_2(\Omega)||); L_2(R^3)||.$$

The most frequently used norm $||\cdot; X^{\gamma, k}_{\infty, 2}||$ is denoted simply by $||\cdot||^{\beta, k}$.

## 2. Basic estimates

From our definitions it follows directly

$$||q; X^{0, k}_{2, 2}|| \leqslant c||q||^{2, k}. \tag{2.1}$$

The following inequalities ([7], theorems 200–202) are used:

$$\left(\int \left(\int q(x, y)\, dy\right)^2 dx\right)^{1/2} \leqslant \int \left(\int q^2(x, y)\, dx\right)^{1/2} dy,$$
$$\left(\int \left(\sum_j q_j(x)\right)^2 dx\right)^{1/2} \leqslant \sum_j \left(\int q_j^2(x)\, dx\right)^{1/2}, \tag{2.2}$$
$$\left(\sum_j \left(\int q_j(x)\, dx\right)^2\right)^{1/2} \leqslant \int \left(\sum_j q_j^2(x)\right)^{1/2} dx$$

which hold for non-negative functions. In view of Eq. (1.2), the following estimate holds

$$c^-_{2.3} w_\lambda \leqslant \nu_{f_0} \leqslant c^+_{2.3} w_\lambda, \tag{2.3}$$

where $\lambda \in [0, 1]$ depends on the particle interaction potential, and $c^-_{2.3}$ and $c^+_{2.3}$ are positive constants, independent of $x$ and $t$.

Let us denote by $\nu_*$ a constant such that

$$0 < \nu_* < c^-_{2.3}. \tag{2.4}$$

We have

$$\left| \frac{\partial^{|\gamma|}}{\partial x^\gamma} f^p_0 \right| \leqslant c w_{2|\gamma|} f^p_0,$$
$$\left| \frac{\partial^{|\gamma|+1}}{\partial x^\gamma \partial t} f^p_0 \right| \leqslant c w_{2(|\gamma|+1)} f^p_0, \tag{2.5}$$
$$\left| \frac{\partial^{|\gamma|}}{\partial x^\gamma} \nu_{f_0} \right| \leqslant c w_\lambda$$

where $\lambda$ is the same as in Eq. (2.3).

GRAD [5] proved that

$$(2.7)\qquad \sup_{\xi\in R^3}\Big|w_\beta(\xi)\int_{R^3}\int_{S^2} k(\xi,\xi_*,\mathfrak{n})M^{1/2}(\xi_*)\cdot\big(M^{1/2}(\xi')q(\xi'_*)$$

$$+M^{1/2}(\xi'_*)q(\xi')+M^{1/2}(\xi)q(\xi_*)\big)d\mathfrak{n}\,d\xi_*\Big| \leqslant c\sup_{R^3}|w_{\beta'},q|,$$

where

$$\beta' = \begin{cases} 0 & \text{for} \quad \beta = 0, \\ \beta-1 & \text{for} \quad \beta = 1,2,\dots, \end{cases}$$

and

$$(2.8)\qquad \sup_{\xi\in R^3}\Big|\int_{R^3}\int_{S^2} k(\xi,\xi_*,\mathfrak{n})M^{1/2}(\xi_*)\big(M^{1/2}(\xi')q(\xi'_*)$$

$$+M^{1/2}(\xi'_*)q(\xi')+M^{1/2}(\xi)q(\xi_*)\big)d\mathfrak{n}\,d\xi_*\Big| \leqslant c\|q;L_2(R^3)\|.$$

Moreover, we have

$$(2.9)\qquad |w_\beta M^{-1/2}J(M^{1/2}q,M^{1/2}r)| \leqslant cw_\lambda(\sup_{R^3}|w_\beta q|)(\sup_{R^3}|w_\beta r|),$$

for $\beta = 0,1,2,\dots$.

## 3. System of CSLE equations

The system of linear CSLE equations has the form

$$\frac{\partial q_0}{\partial t}+\xi\,\mathrm{grad}_x q_0+\frac{1}{\varepsilon}\nu_{f_0}q_0 = \frac{1}{\varepsilon}f_0^{-1/2}\mathscr{K}(f_0^{1/2}q_0)+\frac{1}{\varepsilon}\sum_{i=1,2}\chi_i f_0^{-1/2}\mathscr{K}(M_+^{1/2}q_i),$$

$$\frac{\partial q_1}{\partial t}+\xi\,\mathrm{grad}_x q_1+\frac{1}{\varepsilon}\nu_{f_0}q_1 = \left(-M_+^{-1/2}\left(\frac{\partial}{\partial t}+\xi\,\mathrm{grad}_x\right)f_0^{1/2}\right)q_0$$

$$(3.1)\qquad +\frac{1}{\varepsilon}(1-\chi_1)M_+^{-1/2}\mathscr{K}(M_+^{1/2}q_1)+\frac{2}{\varepsilon}M_+^{-1/2}J(\tilde{f}_0,M_+^{1/2}q_1)+A(f_0^{1/2}q_0)$$

$$+\sum_{i=1,2}A(M_+^{1/2}q_i)+\varepsilon^a\mathscr{A},$$

$$\frac{\partial q_2}{\partial t}+\xi\,\mathrm{grad}_x q_2+\frac{1}{\varepsilon}\nu_{f_0}q_2 = \frac{1}{\varepsilon}(1-\chi_2)M_+^{-1/2}\mathscr{K}(M_+^{1/2}q_2)$$

$$+\frac{2}{\varepsilon}M_+^{-1/2}J(\tilde{f}_0,M_+^{1/2}q_2)+\frac{2}{\varepsilon}M_+^{-1/2}J(\tilde{f}_0,f_0^{1/2}q_0).$$

for $t\in[t_1,t_2]$, $0\leqslant t_1<t_2\leqslant t_0$, with the initial data

$$(3.2)\qquad q_i|_{t=t_1} = Q_i \quad (i = 0,1,2).$$

$\varepsilon\in\,]0,\varepsilon_0]$ is a small parameter representing the mean free path of the mean collision time. $f_0 = f_0(t,x,\xi)$ is a local Maxwellian such that (A1.1) holds. $\mathscr{K}$ is the operator defined by $\mathscr{K}q = 2J_+(f_0, q) - f_0 \cdot \nu_q$. $M_+ = M_+(\xi)$ is a global Maxwellian given by (1.2).

$\tilde{f}_0 = \tilde{f}_0(t/\varepsilon, x, \xi)$ is a function such that:

(A3.1) $\tau \to M_+^{-1/2}\tilde{f}_0(\tau)$ is a continuously differentiable function from $[0, +\infty[$ into $X^{\beta,k}_{\infty,\infty}$ (i.e. $M_+^{-1/2}\tilde{f}_0 \in C^1([0, +\infty[; X^{\beta,k}_{\infty,\infty}))$ where $\beta$ and $k$ are suitably large, and

$$\left\| M_+^{-1/2}\tilde{f}_0\left(\frac{t}{\varepsilon}\right); X^{\beta,k}_{\infty,\infty} \right\| \leqslant \theta \exp\left(-\frac{t}{\varepsilon}\delta\right), \tag{3.3}$$

where the constant $\theta$ can be chosen small enough and the decay exponent $\delta$ is positive.

Next, $a$ is an integer. $\chi_i = \chi_i(\xi)$ is the characteristic function of the ball of radius $\varkappa_i$ with the center at the origin in $R^3$. The operator $A$ is given by $Aq = M_+^{-1/2}J(\mathscr{F}, q)$ where

(A3.2) $M_+^{-1/2}\mathscr{F} \in C^1([0, t_0]; X^{\beta,k}_{\infty,\infty})$

with $\beta$ and $k$ the same as in (A3.1).

In the CSLE linear system ([9]), the equivalent of the sum of nonlinear and nonhomogeneous terms is the term $\mathscr{A}$ which is treated as a given function of $t$. The conditions assumed concerning $\mathscr{A}$ will be specified later.

Putting aside the collisions, we introduce the idea of a (extrinsic or free-streaming) trajectory (see [10]). For fixed $t$ and $\xi$ let $\varphi_{(t,\xi)}$ be the translation on $\Omega$ defined by

$$\varphi_{(t,\xi)}x = x + t\xi \pmod 1 \tag{3.4}$$

If a particle has position $x \in \Omega$ and velocity $\xi$ at time 0, then its position at time $t$, taking no account of collisions, is given by $\varphi_{(t,\xi)}x$. Its (extrinsic) trajectory in $R^1 \times \Omega \times R^3$ is the curve defined parametrically as follows

$$t \to (t, \varphi_{(t,\xi)}x, \xi). \tag{3.5}$$

Now, let $\Phi_t$ be a one-parameter family of operators

$$(\Phi_t q)(x, \xi) = q(\varphi_{(t,\xi)}x, \xi). \tag{3.6}$$

Let us define a function $q^\#$ as the function $q$ considered along the (extrinsic) trajectories (cf. [8]). More precisely,

$$q^\#(t) = \Phi_t q(t). \tag{3.7}$$

An important property of the norms introduced in Sect. 1 is

$$\|q^\#(t)\| = \|q(t)\|. \tag{3.8}$$

Let us now introduce the following two-parameter families of operators

$$U_\varepsilon(t, \sigma)q = (\Phi_{\sigma-t}q)\exp\left(-\frac{1}{\varepsilon}\int_\sigma^t \Phi_{\sigma'-t}\nu_{f_0}(\sigma')d\sigma'\right) \tag{3.9}$$

and

$$V_\varepsilon(t, \sigma)q = q \cdot \exp\left(-\frac{1}{\varepsilon}\int_\sigma^t \nu^\#_{f_0}(\sigma')d\sigma'\right). \tag{3.10}$$

In this paper the following two (equivalent) integral versions of CSLE with initial data (3.2) are analysed

$$(3.11)\qquad q_i(t) = U_\varepsilon(t, t_1)Q_i + \int\limits_{t_1}^{t} U_\varepsilon(t, \sigma)\mathfrak{Q}_i[q_0, q_1, q_2, \mathscr{A}; \varepsilon](\sigma)d\sigma, \qquad i = 0, 1, 2;$$

and

$$(3.12)\qquad q_i^{\#}(t) = V_\varepsilon(t, t_1)(\Phi_{t_1}Q_i) + \int\limits_{t_1}^{t} V_\varepsilon(t, \sigma)(\mathfrak{Q}_i[q_0, q_1, q_2, \mathscr{A}; \varepsilon])^{\#}(\sigma)d\sigma, \qquad i = 0, 1, 2;$$

where $\mathfrak{Q}_0, \mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are the right-hand sides of Eqs. $(3.1)_1$, $(3.1)_2$ and $(3.1)_3$, respectively.

Let us define the following space of functions which depend on $t \in [t_1, t_2]$, $0 \leqslant t_1 < t_2 \leqslant t_0$:

$$Z^{\beta,k} = \{q: q \in L_\infty([t_1, t_2]; X^{\beta,k}_{\infty,2}) \cap C^0([t_1, t_2]; X^{\beta-1,k-1}_{\infty,2}) \cap C^1([t_1, t_2]; X^{\beta-2,k-2}_{\infty,2}) \quad \text{and} \quad q^{\#} \in C^0([t_1, t_2]; X^{\beta-1,k}_{\infty,2}) \cap C^1([t_1, t_2]; X^{\beta-2,k}_{\infty,2})\}$$

and the following norm

$$(3.13)\qquad |||q|||^{\beta,k} = \sup_{t\in[t_1,t_2]} ||q(t)||^{\beta,k}.$$

Let us notice at the end of this section that the operator $\xi \cdot \operatorname{grad}_x + \frac{1}{\varepsilon}\nu_{f_0}$ cannot possess a dense domain in the spaces $X^{\beta,k}_{\infty,2}$. Therefore, although $U_\varepsilon$ is given by a simple expression, it is not continuous with respect to $t$ in such spaces. Neither is $V_\varepsilon$ in the general hard potential cases. Nevertheless, applying the methods known from [11], we are able to prove that a solution to the system (3.11) belongs to $Z^{\beta,k}$.

## 4. Estimates of terms of the integral form of CSLE

Immediately from (1.2), (2.3), (2.4) and (2.5) we obtain

LEMMA 1. Let $0 \leqslant \sigma \leqslant t$. Then

$$\left|\frac{\partial^{|\gamma|}}{\partial x^\gamma}\exp\left(-\frac{1}{\varepsilon}\int\limits_{\sigma}^{t}\Phi_{\sigma'-t}\nu_{f_0}(\sigma')d\sigma'\right)\right| \leqslant c\exp\left(-\frac{t-\sigma}{\varepsilon}\nu_* w_\lambda\right)$$

for all $\gamma$, where constant $c$ depends on $|\gamma|$. □

LEMMA 2.

(i) $$\left|\left|\left|\int\limits_{t_1}^{t} U_\varepsilon(t, \sigma)\mathfrak{Q}_0[q_0, q_1, q_2; \varepsilon](\sigma)d\sigma\right|\right|\right|^{\beta,k} \leqslant c|||q_0|||^{\beta',k} + \sum_{i=1,2} c\exp(c\varkappa_i^2)\,|||q_i|||^{0,k},$$

where $\beta'$ is the same as in (2.7);

(ii) $$\left|\left|\left|\int\limits_{t_1}^{t}\frac{1}{\varepsilon}U_\varepsilon(t, \sigma)\left(f_0^{-1/2}\mathscr{K}(f_0^{1/2}q_0)\right)(\sigma)d\sigma\right|\right|\right|^{0,k} \leqslant c\sup_{[t_1,t_2]}||q_0; X^{0,k}_{2,2}||;$$

(iii) $$\left|\left|\left| \int_{t_1}^{t} U_\varepsilon(t,\sigma)\mathfrak{Q}_0[q_0,q_1,q_2;\varepsilon](\sigma)d\sigma \right|\right|\right|^{\beta,k}$$

$$\leqslant c\,\frac{t_2-t_1}{\varepsilon}\,|||q_0|||^{\beta',k}+\sum_{i=1,2} c\exp(c\varkappa_i^2)\,\frac{t_2-t_1}{\varepsilon}\,|||q_i|||^{0,k}.$$

P r o o f. First, we investigate the term

$$I_{4,1}=\int_{t_1}^{t}\frac{1}{\varepsilon}\,U_\varepsilon(t,\sigma)\big(f_0^{-1/2}\mathscr{K}(f_0^{1/1}q_0)\big)(\sigma)d\sigma. \tag{4.1}$$

Let us notice that

$$f_0^{-1/2}(t,\varphi_{(\sigma-t,\xi)}x,\xi)\int_{R^3}\int_{S^2}k(\xi,\xi_*,\mathfrak{n})f_0(t,\varphi_{(\sigma-t,\xi)}x,\xi')$$

$$\cdot f_0^{1/2}(t,\varphi_{(\sigma-t,\xi)}x,\xi'_*)q_0(t,\varphi_{(\sigma-t,\xi)}x,\xi'_*)d\mathfrak{n}\,d\xi_*=\int_{R^3}\int_{S^2}k(\xi,\xi_*,\mathfrak{n})f_0^{1/2}(t,\varphi_{(\sigma-t,\xi)}x,\xi')$$

$$\cdot f_0^{1/2}(t,\varphi_{(\sigma-t,\xi)}x,\xi_*)q_0(t,\varphi_{(\sigma-t,\xi)}x,\xi'_*)d\mathfrak{n}\,d\xi_*.$$

Using Eqs. (2.5) and (1.2), the $\dfrac{\partial^{|\gamma|}}{\partial x^\gamma}$ derivatives of the last term can be estimated by

$$c\sum_{0\leqslant|\gamma'|\leqslant|\gamma|}\int_{R^3}\int_{S^2}k(\xi,\xi_*,\mathfrak{n})M_+^{1/2}(\xi')M_+^{1/2}(\xi_*)\frac{\partial^{|\gamma'|}}{\partial x^{\gamma'}}q_0(t,\varphi_{(\sigma-t,\xi)}x,\xi'_*)d\mathfrak{n}\,d\xi_*.$$

The other terms in $f_0^{-1/2}\mathscr{K}(f_1^{1/2}q_0)$ are estimated in the same way. Thus, by Eq. (2.2) and Lemma 1 we have

$$||I_{4\cdot1}||^{\beta,k}_{\_}\leqslant c\sum_{0\leqslant|\gamma|\leqslant k}\int_{t_1}^{t}\frac{1}{\varepsilon}\exp\left(-\frac{t-\sigma}{\varepsilon}\nu_*\right) \tag{4.2}$$

$$\cdot\sup_{\xi\in R^3}\left(w_\beta(\xi)\int_{R^3}\int_{S^2}k(\xi,\xi_*,\mathfrak{n})M_+^{1/2}(\xi_*)\cdot\left(M_+^{1/2}(\xi')\left\|\frac{\partial^{|\gamma|}}{\partial x^\gamma}q_0(\sigma,\cdot,\xi'_*);L_2(\Omega)\right\|\right.\right.$$

$$\left.\left.+M_+^{1/2}(\xi'_*)\left\|\frac{\partial^{|\gamma|}}{\partial x^\gamma}q_0(\sigma,\cdot,\xi');L_2(\Omega)\right\|+M_+^{1/2}(\xi)\left\|\frac{\partial^{|\gamma|}}{\partial x^\gamma}q_0(\alpha,\cdot,\xi_*);L_2(\Omega)\right\|\right)d\mathfrak{n}\,d\xi_*\right)d\sigma.$$

From Eq. (2.7) we obtain

$$|||I_{4.1}|||^{\beta,k}\leqslant c|||q_0|||^{\beta',k}. \tag{4.3}$$

Similarly, by means of Eq. (2.8), we obtain (ii). Next, let us investigate the term

$$I_{4.4}=\int_{t_1}^{t}\frac{1}{\varepsilon}\,U_\varepsilon(t,\sigma)\big(\chi_i f_0^{-1/2}\mathscr{K}(M_+^{1/2}q_i)\big)(\sigma)d\sigma. \tag{4.4}$$

We notice that

$$(4.5)\qquad \left| w_\beta \chi_i \frac{\partial^{|\gamma|}}{\partial x^\gamma}(f_0^{-1/2}M_+^{1/2})\right| \leqslant c w_{\beta+2|\gamma|}\chi_i f_0^{-1/2}M_+^{1/2} \leqslant c\exp(c\varkappa_i^2)$$

and the term $M_+^{-1/2}\mathscr{K}(M_+^{1/2}q_i)$ can be estimated in the same way as previously. Therefore we obtain (i).

Next, we can estimate $\exp\left(-\frac{t-\sigma}{\varepsilon}\nu_*\right)$ by 1 in (4.2) and (4.4) to obtain (iii). □

LEMMA 3.

(i) $||\chi_i f_0^{-1/2}\mathscr{K}(M_+^{1/2}q_i); X_{2;2}^{0;2}|| \leqslant c\exp(c\varkappa_i^2)||q_i||^{0,k}$;

(ii) $\left\|\frac{\partial}{\partial x_i}(2f_0^{-1/2}J(f_0, f_0^{1/2}q_0)) - 2f_0^{-1/2}J\left(f_0, f_0^{1/2}\frac{\partial q_0}{\partial x_i}\right); X_{2,2}^{0,0}\right\| \leqslant c||q_0||^{2+\lambda,0}$,

where $\lambda$ is the same as in Eq. (2.3).

P r o o f. (i) follows from Eqs. (2.1), (4.5) and (2.7). To prove (ii) let us notice that

$$(4.6)\qquad \frac{\partial}{\partial x_i}(f_0^{-1/2}\mathscr{K}(f_0^{1/2}q_0)) - f_0^{-1/2}\mathscr{K}\left(f_0^{1/2}\frac{\partial q_0}{\partial x_i}\right)$$

$$= \int_{R^3}\int_{S_1} k(\xi,\xi_*,\mathfrak{n})\left\{\frac{\partial}{\partial x_i}(f_0^{1/2}(\xi')\cdot f_0^{1/2}(\xi_*))q_0(\xi'_*)\right.$$

$$\left. + \frac{\partial}{\partial x_i}(f_0^{1/2}(\xi'_*)f_0^{1/2}(\xi_*))q_0(\xi') - \frac{\partial}{\partial x_i}(f_0^{1/2}(\xi)f_0^{1/2}(\xi_*))q_0(\xi_*)d\mathfrak{n}\,d\xi_*\right.$$

and

$$(4.7)\qquad \frac{\partial}{\partial x_i}(q_0\nu_{f_0}) - \frac{\partial q_0}{\partial x_i}\nu_{f_0} = q_0\frac{\partial}{\partial x_i}\nu_{f_0}.$$

Using Eq. (2.1), applying the methods of Lemma 2 to (4.6) and using Eqs. (2.6) to (4.7) we obtain (ii). □

LEMMA 4.

$$\left|\left|\left| \int_{t_1}^{t} U_\varepsilon(t,\sigma)\mathfrak{Q}_1[q_0,q_1,q_2,\mathscr{A};\varepsilon](\sigma)d\sigma\right|\right|\right|^{\beta,k} \leqslant c\varepsilon|||q_0|||^{0,k}$$

$$+\left(\frac{c}{(1+\varkappa_1)^{1+\lambda}} + c\theta + c\varepsilon\right)|||q_1|||^{\beta,k} + c\varepsilon|||q_2|||^{\beta,k} + c\varepsilon^{a+1}\left|\left|\left|\frac{1}{w_\lambda}\mathscr{A}\right|\right|\right|^{\beta,k}.$$

P r o o f. By Eqs. (2.5) and (1.2) we have

$$(4.8)\qquad \left| w_\beta M_+^{-1/2}\frac{\partial^{|\gamma|}}{\partial x^\gamma}\left(\frac{\partial f_0^{1/2}}{\partial t} + \xi\cdot \operatorname{grad}_x f_0^{1/2}\right)\right| \leqslant c.$$

Therefore, using (2.2) and Lemma 1 we obtain

$$(4.9)\qquad \left|\left|\left| \int_{t_1}^{t} U_\varepsilon(t,\sigma)\left(-M_+^{-1/2}\left(\frac{\partial f_0^{1/2}}{\partial t} + \xi\cdot\operatorname{grad}_x f_0^{1/2}\right)q_0\right)(\sigma)d\sigma\right|\right|\right|^{\beta,k} \leqslant c\varepsilon|||q_0|||^{0,k}.$$

Next, we have

$$\sup_{\xi}\left((1-\chi_1)\frac{1}{w_{1+\lambda}}\int_{t_1}^{t}\frac{w_\lambda}{\varepsilon}\exp\left(-\frac{t-\sigma}{\varepsilon}\nu_* w_\lambda\right)d\sigma\right) \leqslant \frac{c}{(1+\varkappa_1)^{1+\lambda}}. \tag{4.10}$$

Thus, by Eq. (2.2), Lemma 1, Eqs. (2.5), (1.2) and (2.7) we obtain

$$\left|\left|\left|\int_{t_1}^{t}\frac{1}{\varepsilon}U_\varepsilon(t,\sigma)\big((1-\chi_1)M_+^{-1/2}\mathscr{K}(M_+^{1/2}q_1)\big)(\sigma)d\sigma\right|\right|\right|^{\beta,k} \leqslant \frac{c}{(1+\varkappa_1)^{1+\lambda}}|||q_1|||^{\beta,k}. \tag{4.11}$$

Finally, by Eqs. (3.3) and (2.9) we have

$$\left|\left|\int_{t_1}^{t}\frac{2}{\varepsilon}U_\varepsilon(t,\sigma)\big(M_+^{-1/2}J(\tilde{f}_0, M_+^{1/2}q_1)\big)(\sigma)d\sigma\right|\right|^{\beta,k} \leqslant c\exp\left(-\frac{t}{\varepsilon}\delta\right)\theta\, |||q_1|||^{\beta,k}. \tag{4.12}$$

Similar arguments and (A3.2) yield the estimates of the other terms. □

In the same way we obtain

LEMMA 5.

(i) $$\left|\left|\left|\int_{t_1}^{t}U_\varepsilon(t,\sigma)\mathfrak{Q}_2[q_0,q_2;\varepsilon](\sigma)\,d\sigma\right|\right|\right|^{\beta,k} \leqslant \left(\frac{c}{(1+\varkappa_2)^{1+\lambda}}+c\theta\right)|||q_2|||^{\beta,k}+c\theta|||q_0|||^{0,k};$$

(ii) $$\left|\left|\left|\int_{t_1}^{t}U_\varepsilon(t,\sigma)\mathfrak{Q}_2[q_0,q_2;\varepsilon](\sigma)d\sigma\right|\right|\right|^{\gamma,k}$$

$$\leqslant \frac{1}{\varepsilon}\frac{c}{1+\varkappa_2}\int_{t_1}^{t}\exp\left(-\frac{t-\sigma}{\varepsilon}\nu_*\right)||q_2(\sigma)||^{\beta,k}d\sigma$$

$$+c\frac{\theta}{\varepsilon}\int_{t_1}^{t}\exp\left(-\frac{t-\sigma}{\varepsilon}\nu_*-\frac{\sigma}{\varepsilon}\delta\right)\big(||q_2(\sigma)||^{\beta+1,k}+||q_0(\sigma)||^{0,k}\big)d\sigma.$$ □

## 5. Solution to the integral form of CSLE

We construct a solution to the integral form of CSLE in the time interval $[t_1, t_2]$, where $t_1$ is given and $t_2$ will be specified below, by the method of successive approximations. Let $q_i^0 = 0$ $(i = 0, 1, 2)$ and $(q_0^j, q_1^j, q_2^j)$ for $j = 1, 2, 3, \ldots$ be given by

$$q_i^j(t) = U_\varepsilon(t,t_1)Q_i + \int_{t_1}^{t}U_\varepsilon(t,\sigma)\mathfrak{Q}_i[q_0^{j-1}, q_1^{j-1}, q_2^{j-1}, \mathscr{A};\varepsilon](\sigma)d\sigma. \tag{5.1}$$

Now, by Lemma 2 (iii), Lemma 4 and Lemma 5 (i), if we choose

(C5.1) $\varkappa_1, \varkappa_2$ — large enough;

(C5.2) $\theta_0, \varepsilon_0$ — positive and small enough ($\theta_0$ — dependent on $\varkappa_1, \varkappa_2$; and $\varepsilon_0$ — dependent on $\varkappa_1, \varkappa_2, \theta_0$);

(C5.3) fixed $\varepsilon$ and $\theta$ such that $0 < \varepsilon \leqslant \varepsilon_0, 0 \leqslant \theta \leqslant \theta_0$;

(C5.4) $t_2$ such that $\frac{t_2 - t_1}{\varepsilon}$ is positive and small enough (dependent on $\varkappa_1$ and $\varkappa_2$)

then we find positive constants $b_0 < 1$, $b_1$ and $b_2$ such that the inequalities

$$\sum_{i=0}^{2} |||q_i^{j+1} - q_i^j|||^{\beta_i, k} \leqslant b_0 \sum_{i=0}^{2} |||q_i^j - q_i^{j-1}|||^{\beta_i, k} \tag{5.2}$$

and

$$\sum_{i=0}^{2} |||q_i^j|||^{\beta_i, k} \leqslant b_1 \sum_{i=0}^{2} ||Q_i||^{\beta_i, k} + b_2 \left|\left|\left| \frac{1}{w_\lambda} \mathscr{A} \right|\right|\right|^{\beta_1, k} \tag{5.3}$$

hold for all integers $j$ and integers $\beta_i$ $(i = 1, 2, 3)$ such that $\beta_2 \geqslant \beta_1$. Thus the sequence $\{(q_0^j, q_1^j, q_2^j)\}$ converges in $\prod_{i=0,1,2} L_\infty([t_1, t_2]; X_{\infty, k}^{\beta_i, k})$. Let its limit be denoted by $(q_0, q_1, q_2)$. We will investigate the differentiability of $(q_0, q_1, q_2)$ in the next Sections. Let us note that $\varepsilon$ will not be important in this consideration. Therefore, we take $\varepsilon = 1$, for simplicity.

## 6. Some lemmas

In agreement with the previous remark, let us put $\varepsilon = 1$. Similarly as in Lemma 1, we have

LEMMA 6. Let $\sigma \leqslant t$. Then

$$\left| \frac{\partial^{|\gamma|}}{\partial x^\gamma} \left( 1 - \exp\left( - \int_\sigma^t \Phi_{\sigma' - t} \nu_{f_0}(\sigma') d\sigma' \right) \right) \right| \leqslant c(t - \sigma) w_\lambda$$

for all $\gamma$, where the constant $c$ depends on $|\gamma|$. □

LEMMA 7. Let $Q \in X_{\infty, k}^{\beta, k}$. Then

(i) $\lim_{h \to 0} ||U(t+h, t_1)Q - U(t, t_1)Q||^{\beta - 1, k - 1} = 0$,

(ii) $$\lim_{h \to 0} \left|\left| \frac{1}{h} \left( U(t+h, t_1)Q - U(t, t_1)Q \right) + \nu_{f_0}(t) \cdot U(t, t_1)Q + \right.\right.$$
$$\left.\left. + \xi \cdot \mathrm{grad}_x U(t, t_1)Q \right|\right|^{\beta - 2, k - 2} = 0, \quad \text{for} \quad t \in [t_1, t_2].$$

P r o o f. For simplicity, let $h > 0$. First, we show the following estimates

$$||\Phi_{t_1 - (t+h)} Q - \Phi_{t_1 - t} Q||^{\beta - 1, k - 1} \leqslant ch \tag{6.1}$$

and

$$\left|\left| \frac{1}{h} (\Phi_{t_1 - (t+h)} Q - \Phi_{t_1 - t} Q) + \xi \cdot \mathrm{grad}_x (\Phi_{t_1 - t} Q) \right|\right|^{\beta - 2, k - 2} \leqslant ch. \tag{6.2}$$

Let $\xi \in R^3$ be fixed and $Q(\cdot, \xi) \in C^k(\Omega)$. Then we have

$$\Phi_{t_1 - (t+h)} Q - \Phi_{t_1 - t} Q = h\xi \cdot \mathrm{grad}_x \Phi_{t_1 - (t + \sigma_0 h)} Q, \tag{6.3}$$

where $\sigma_0 \in [0, 1]$, and

$$(6.4)\qquad \frac{1}{h}(\Phi_{t_1-(t+h)}Q-\Phi_{t_1-t}Q)+\xi\cdot\operatorname{grad}_x(\Phi_{t_1-t}Q) = \frac{h}{2}(\xi\cdot\operatorname{grad}_x)^2(\Phi_{t_1-(t-\sigma_1 h)}Q),$$

where $\sigma_1 \in ]0, 1[$.

Approximating $H_2^k(\Omega)$-functions by $C^k(\Omega)$-functions (cf. [4] — Part 1, Lemma 15.1) and using the fact that $\Omega$ is a torus (cf. [1] — Theorem 3.14) we obtain

$$(6.5)\qquad \|\Phi_{t_1-(t+h)}Q-\Phi_{t_1-t}Q;\, H_2^{k-1}(\Omega)\| \leqslant chw_1\|Q;\, H_2^k(\Omega)\|$$

and

$$(6.6)\qquad \left\|\frac{1}{h}(\Phi_{t_1-(t+h)}Q-\Phi_{t_1-t}Q)+\xi\cdot\operatorname{grad}_x(\Phi_{t_1-t}Q);\, H_2^{k-2}(\Omega)\right\| \leqslant chw_2\|Q;\, H_2^k(\Omega)\|,$$

for $\xi \in R^3$. Hence, Eqs. (6.1) and (6.2) follow. Next, owing to the Assumption (A1.1), the function $\nu_{f_0}(\cdot,\cdot,\xi)$, for fixed $\xi$ is smooth (in the classical sense). Therefore, using Lemma 1 we obtain (i) and (ii) in the same way as previously. □

LEMMA 8. Let $Q \in X^{\beta,k}_{\infty,2}$. Then

$$\text{(i)}\qquad \lim_{h\to 0}\|V(t+h,t_1)(\Phi_{t_1}Q)-V(t,t_1)(\Phi_{t_1}Q)\|^{\beta-1,k} = 0$$

and

$$\text{(ii)}\qquad \lim_{h\to 0}\left\|\frac{1}{h}\big(V(t+h,t_1)(\Phi_{t_1}Q)-V(t,t_1)(\Phi_{t_1}Q)\big)+\nu^{\ddagger}_{f_0}(t)\cdot V(t,t_1)(\Phi_{t_1}Q)\right\|^{\beta-2,k} = 0$$

for $t \in [t_1, t_2]$.

Proof. Let $h > 0$, for simplicity. We have

$$(6.7)\qquad V(t+h,t_1)(\Phi_{t_1}Q)-V(t,t_1)(\Phi_{t_1}Q) = V(t,t_1)(\Phi_{t_1}Q)\left(\exp\left(-\int_t^{t+h}\nu^{\ddagger}_{f_0}(\sigma')\,d\sigma'\right)-1\right).$$

Hence (i) and (ii) follow immediately by the same arguments as previously. □

LEMMA 9. Let $\frac{1}{w_\lambda}\mathfrak{Q} \in L_\infty([t_1,t_2]; X^{\beta,k}_{\infty,2})\cap C^0([t_1,t_2]; X^{\beta-1,k-1}_{\infty,2})$.

Then

$$\text{(i)}\qquad \lim_{h\to 0}\left\|\int_{t_1}^{t+h}U(t+h,\sigma)\mathfrak{Q}(\sigma)\,d\sigma-\int_{t_1}^{t}U(t,\sigma)\mathfrak{Q}(\sigma)\,d\sigma\right\|^{\beta-1,k-1} = 0$$

and

$$\text{(ii)}\qquad \lim_{h\to 0}\left\|\frac{1}{h}\left(\int_{t_1}^{t+h}U(t+h,\sigma)\mathfrak{Q}(\sigma)\,d\sigma-\int_{t_1}^{t}U(t,\sigma)\mathfrak{Q}(\sigma)\,d\sigma\right)\right.$$

$$\left.+\nu_{f_0}(t)\cdot\int_{t_1}^{t}U(t,\sigma)\mathfrak{Q}(\sigma)\,d\sigma+\xi\cdot\operatorname{grad}_x\int_{t_1}^{t}U(t,\sigma)\mathfrak{Q}(\sigma)\,d\sigma-\mathfrak{Q}(t)\right\|^{\beta-2,k-2}$$

$$= 0 \qquad \text{for} \qquad t \in [t_1, t_2]$$

Proof. Let $h > 0$. We have

(6.8) $$\int_{t_1}^{t+h} U(t+h,\sigma)\mathfrak{Q}(\sigma)d\sigma - \int_{t_1}^{t} U(t,\sigma)\mathfrak{Q}(\sigma)d\sigma$$
$$= \int_0^h U(t+h,t+\sigma)\mathfrak{Q}(t+\sigma)d\sigma + \int_{t_1}^{t} \big(U(t+h,\sigma)\mathfrak{Q}(\sigma) - U(t,\sigma)\mathfrak{Q}(\sigma)\big)d\sigma.$$

Now let us examine the first term of the right-hand side of Eq. (6.8). We have

(6.9) $$\left\|\int_0^h U(t+h,t+\sigma)\mathfrak{Q}(t+\sigma)d\sigma\right\|^{\beta-1,k-1} \leqslant \mathrm{ch}\left|\left|\left|\frac{1}{w_\lambda}\mathfrak{Q}\right|\right|\right|^{\beta,k-1}$$

and

(6.10) $$\left\|\frac{1}{h}\int_0^h U(t+h,t+\sigma)\mathfrak{Q}(t+\sigma)d\sigma - \mathfrak{Q}(t)\right\|^{\beta-2,k-2}$$
$$\leqslant \frac{1}{h}\int_0^h \left\|\frac{1}{w_\lambda}\big(\mathfrak{Q}(t+\sigma)-\mathfrak{Q}(t)\big)\right\|^{\beta-1,k-2} d\sigma + \frac{1}{h}\int_0^h \left\|\frac{1}{w_\lambda}\big((\Phi_{\sigma-h}\mathfrak{Q}(t+\sigma)-\mathfrak{Q}(t+\sigma)\big)\right\|^{\beta-1,k-2} d\sigma$$
$$+\left|\left|\left|\frac{1}{w_\lambda}\mathfrak{Q}\right|\right|\right|^{\beta,k-2}\cdot\frac{1}{h}\int_0^h \sup_{\substack{x\in\Omega\\ \xi\in R^3\\ 0\leqslant|\gamma|\leqslant k-2}} \left|\frac{1}{w_\lambda}\frac{\partial^{|\gamma|}}{\partial x^\gamma}\left(1-\exp\left(-\int_\sigma^h \Phi_{\sigma'-h}\nu_{f_0}(t+\sigma')d\sigma'\right)\right)\right| d\sigma.$$

Due to $\frac{1}{w_\lambda}\mathfrak{Q} \in C^0([t_1,t_2]; X_{\infty,2}^{\beta-\frac{1}{2},k-1})$, the first term of the right-hand side of Eq. (6.10) tends to 0 with $h$. By the same arguments as in Lemma 7 we conclude that the second term of the right-hand side of (6.10) can be estimated by

(6.11) $$ch\left|\left|\left|\frac{1}{w_\lambda}\mathfrak{Q}\right|\right|\right|^{\beta,k-1}.$$

By Lemma 6, the third term is estimated by

(6.11′) $$ch\left|\left|\left|\frac{1}{w_\lambda}\mathfrak{Q}\right|\right|\right|^{\beta,k-2}.$$

Next, we examine the second term of the right-hand side of Eq. (6.8). Using the arguments as in Lemmas 7 and 4 we obtain

(6.12) $$\lim_{h\to 0}\left\|\int_{t_1}^{t}\big(U(t+h,\sigma)\mathfrak{Q}(\sigma) - U(t,\sigma)\mathfrak{Q}(\sigma)\big)d\sigma\right\|^{\beta-1,k-1} = 0$$

and

$$(6.13)\quad \lim_{h\leftarrow 0}\left\| \frac{1}{h}\int_{t_1}^{t} \big(U(t+h,\sigma)\mathfrak{Q}(\sigma)-U(t,\sigma)\mathfrak{Q}(\sigma)\big)d\sigma + \nu_{f_0}(t)\int_{t_1}^{t} U(t,\sigma)\mathfrak{Q}(\sigma)d\sigma + \xi\cdot \mathrm{grad}_x \int_{t_1}^{t} U(t,\sigma)\mathfrak{Q}(\sigma)d\sigma \right\|^{\beta-2,k-2} = 0.$$

To end the proof let us note that the same results can be obtained for $h < 0$. □

In the same way we obtain the following lemma.

LEMMA 10. Let $\frac{1}{w_\lambda}\mathfrak{Q}^{*} \in L_\infty([t_1,t_2]; X^{\beta,k}_{\infty,2}) \cap C^0([t_1,t_2]; X^{\beta-1,k}_{\infty,2})$.

Then

$$\text{(i)}\quad \lim_{h\to 0}\left\| \int_{t}^{t+h} V(t+h,\sigma)\mathfrak{Q}^{*}(\sigma)d\sigma - \int_{t_1}^{t} V(t,\sigma)\mathfrak{Q}^{*}(\sigma)d\sigma \right\|^{\beta-1,k} = 0$$

and

$$\text{(ii)}\quad \lim_{h\to 0}\left\| \frac{1}{h}\left(\int_{t_1}^{t+h} V(t+h,\sigma)\mathfrak{Q}^{*}(\sigma)d\sigma - \int_{t_1}^{t} V(t,\sigma)\mathfrak{Q}^{*}(\sigma)d\sigma\right) + \nu^{*}_{f_0}(t)\cdot \int_{t_1}^{t} V(t,\sigma)\mathfrak{Q}^{*}(\sigma)d\sigma - \mathfrak{Q}^{*}(t) \right\|^{\beta-2,k} = 0$$

for $t \in [t_1, t_2]$. □

A simple consequence of Lemmas 7, 8, 9 and 10 is the following lemma.

LEMMA 11. Let $\beta \geqslant 2$, $k \geqslant 2$, $Q \in X^{\beta,k}_{\infty,2}$, $\frac{1}{w_\lambda}\mathfrak{Q} \in Z^{\beta,k}$

and

$$(6.14)\qquad q(t) = U(t,t_1)Q + \int_{t_1}^{t} U(t,\sigma)\mathfrak{Q}(\sigma)d\sigma, \qquad t\in[t_1,t_2].$$

Then

(i) $q \in Z^{\beta,k}$;

(ii) $q$ is a strong solution in $X^{\beta-2,k-2}_{\infty,2}$ of the problem

$$(6.15)\qquad \frac{\partial q}{\partial t} + \xi\cdot \mathrm{grad}_x q + \nu_{f_0}\cdot q = \mathfrak{Q}, \qquad q|_{t=t_1} = Q;$$

and

(iii) $q^{*}$ is a solution in $X^{\beta-2,k}_{\infty,2}$ of the problem

$$(6.16)\qquad \frac{\partial q^{*}}{\partial t} + \nu^{*}_{f_0}\cdot q^{*} = \mathfrak{Q}^{*}, \qquad q^{*}|_{t=t_1} = \Phi_{t_1}Q.$$

## 7. Main result

We now return to the sequence $\{(q_0^j, q_1^j, q_2^j)\}$ given by Eq. (5.1). We have proved that it converges in $\prod_{i=0,1,2} L_\infty([t_1, t_2]; X_{\infty,2}^{\beta_i,k})$ to $(q_0, q_1, q_2)$ provided (C5.1)–(C5.4) hold. Let $\beta_0 \geqslant 2$, $\beta_1 \geqslant 2$, $\beta_2 \geqslant \beta_1$ and $k \geqslant 2$. Then, by Lemma 11, if

$$Q_i \in X_{\infty,2}^{\beta_i,k} \quad (i = 0, 1, 2), \quad \frac{1}{w_\lambda} \mathscr{A} \in Z^{\beta_1,k} \quad \text{and} \quad q_i^{j-1} \in Z^{\beta_i,k}$$

$(i = 0, 1, 2; j = 1)$ then

(i) $q_i^j \in Z^{\beta_i,k}$;

(ii) $(q_0^j, q_1^j, q_2^j)$ is a strong solution in $X_{\infty,2}^{\beta_i-2,k-2}$ of the system

$$\frac{\partial q_i^j}{\partial t} + \xi \cdot \mathrm{grad}_x q_i^j + \nu_{f_0} \cdot q_i^j = \mathfrak{Q}_i[q_0^{j-1}, q_1^{j-1}, q_2^{j-1}, \mathscr{A}], \tag{7.1}$$

$$q_i^j|_{t=t_1} = Q_i, \quad i = 0, 1, 2;$$

and

(iii) $(q_0^j, q_1^j, q_2^j)$ is a solution in $X_{\infty,2}^{\beta_i-2,k}$ of the system

$$\frac{\partial q_i^{j\#}}{\partial t} + \nu_{f_0}^\# \cdot q_i^{j\#} = (\mathfrak{Q}_i[q_0^{j-1}, q_1^{j-1}, q_2^{j-1}, \mathscr{A}])^\#, \tag{7.2}$$

$$q_i^{j\#}|_{t=t_1} = \Phi_{t_1} Q_i.$$

Thus

$$q_i \in C^0([t_1, t_2]; X_{\infty,2}^{\beta_i-1,k-1}), \quad q_i^\# \in C^0([t_1, t_2]; X_{\infty,2}^{\beta_i-1,k})$$

and the sequences $\{\xi \cdot \mathrm{grad}_x q_i^j\}$, $\{\nu_{f_0} \cdot q_i^j\}$, $\{\mathfrak{Q}_i[q_0^{j-1}, q_1^{j-1}, q_2^{j-1}, \mathscr{A}]\}$ converge to $\xi \cdot \mathrm{grad}_x q_i$, $\nu_{f_0} \cdot q_i$, $\mathfrak{Q}_i[q_0, q_1, q_2, \mathscr{A}]$, respectively, in $C^0([t_1, t_2]; X_{\infty,2}^{\beta_i-2,k-2})$; moreover, the sequences $\{\nu_{f_0}^\# \cdot q_i^{j\#}\}$ and $\{(\mathfrak{Q}_i[q_0^{j-1}, q_1^{j-1}, q_2^{j-1}, \mathscr{A}])^\#\}$ converge to $\nu_{f_0}^\# \cdot q_i^\#$ and $(\mathfrak{Q}_i[q_0, q_1, q_2, \mathscr{A}])^\#$ respectively, in $C^0([t_1, t_2]; X_{\infty,2}^{\beta_i-2,k})$. Thus $\left\{\frac{\partial q_i^j}{\partial t}\right\}$ and $\left\{\frac{\partial q_i^j}{\partial t}\right\}^\#$ converge to $\mathfrak{Q}_i[q_0, q_1, q_2; \mathscr{A}] - \nu_{f_0} \cdot q_i - \xi \cdot \mathrm{grad}_x q_i$ and $(\mathfrak{Q}_i[q_0, q_1, q_2; \mathscr{A}])^\# - \nu_{f_0}^\# \cdot q_i^\#$, respectively, in the suitable spaces. In this way we obtain that $q_i(t)$ and $q_i^\#(t)$ are continuously differentiable in $X_{\infty,2}^{\beta_i-2,k-2}$ and $X_{\infty,2}^{\beta_i-2,k}$, respectively. This completes the proof of the following theorem:

THEOREM. *Let $\beta_0 \geqslant 2$, $\beta_1 \geqslant 2$, $\beta_2 \geqslant \beta_1$, $k \geqslant 2$ and $0 \leqslant t_1 < t_0$. Let the constants $\varepsilon$, $\theta$ and $t_2$ be the same as in* (C5.3) *and* (C5.4). *Furthermore, let the Assumptions* (A1.1), (A3.1) (*with $\beta = \beta_2$*) *and* (A3.2) (*with $\beta = \beta_1$*) *be satisfied and*

$$Q_i \in X_{\infty,2}^{\beta_i,k} \quad (i = 0, 1, 2), \quad \frac{1}{w_\lambda} \mathscr{A} \in Z^{\beta_1,k},$$

*Then there exists such $(q_0, q_1, q_2) \in \prod_{i=0,1,2} Z^{\beta_i,k}$ which is a unique, strong in $\prod_{i=0,1,2} X_{\infty,2}^{\beta_i-2,k-2}$ solution of* CSLE (3.1) *with the initial data $q_i|_{t=0} = Q_i$.*

This result is used to prove the existence of a solution of CSLE in the whole time interval $[0, t_0]$ and to obtain suitable estimates of this solution. Namely, in [9] we have proved the a priori estimates, provided that the objects considered are smooth in the sense mentioned above. In this paper the existence of solution and its smoothness have been proved only in the small time interval. However, the a priori estimates taken from [9] make it possible to obtain a solution of CSLE in the whole time interval $[0, t_0]$, by the continuation arguments. Furthermore, the fulfillment of the a priori estimates on $[0, t_0]$ enables us to treat the full nonlinear problem by the method of successive approximations (see [9]).

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WARSAW UNIVERSITY
INSTITUTE OF MECHANICS.

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