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Homogenization of Cosserat continuum

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THE PROBLEM of the homogenization of Cosserat continuum with a periodic structure is studied by means of the energy method. Homogenized quantities are derived and correctors are introduced. An example of the determination of homogenized quantities for the one-dimensional case is also presented.

Przeprowadzono homogenizację ośrodka Cosseratów o strukturze periodycznej na podstawie metody energetycznej. Otrzymano stałe materiałowe dla ośrodka zhomogenizowanego, który jest także ośrodkiem Cosseratów. Wyprowadzono postać tzw. korektorów. Podano zamkniętą postać stałych materiałowych zhomogenizowanych dla przypadku warstwowej struktury periodycznej.

Проведена гомогенизация среды Коссера с периодической структурой опираясь на энергетический метод. Получены материальные постоянные для гомогенизованной среды, которая также является средой Коссера. Выведен вид т. наз. коректоров. Приведен замкнутый вид гомогенизированных материальных постоянных для случая слоистой периодической структуры.

1. Introduction

HOMOGENIZATION is a method which consists in replacing a heterogeneous body by an equivalent homogeneous body. The procedure of homogenization has precise mathematical foundations [1, 2, 9]. For media with periodic and quasi-periodic structures it is possible to determine effectively average quantities. There are many papers which are related to homogenization of elastic continuum [3, 4, 5, 7].

In this paper we shall investigate Cosserat medium for which all material coefficients are assumed to be periodic functions. Section 2 of this paper is concerned with the variational (weak) formulation of the boundary value problem for Cosserat medium. In Sect. 3 the homogenization problem is formulated and resolved by means of the energy method. In Sect. 4 correctors are introduced and studied. An example of the calculation of the homogenized quantities for the one-dimensional case is presented in Sect. 5.

2. Equations of Cosserat continuum

In this section we shall formulate the basic equations describing Cosserat medium [8] in a form suitable for later homogenization. We follow [6] where the existence and uniqueness of the weak solution of the boundary value problem for Cosserat continuum is studied.

We write the equations of static equilibrium

(2.1) $$m_{ji,j}+\varepsilon_{ijk}\tau_{jk}+Y_i = 0,$$

(2.2) $$\tau_{ji,j}+X_i = 0,$$

geometrical equations

(2.3) $$\gamma_{ij} = u_{j,i}-\varepsilon_{ijk}\varphi_k,$$

(2.4) $$\varkappa_{ij} = \varphi_{j,i},$$

and constitutive relations for an anisotropic inhomogeneous material

(2.5) $$\tau_{ij} = E_{ijkl}\gamma_{kl}+K_{ijkl}\varkappa_{kl},$$

(2.6) $$m_{ij} = K_{klij}\gamma_{kl}+M_{ijkl}\varkappa_{kl},$$

with

(2.7) $$E_{ijkl} = E_{klij}, \quad M_{ijkl} = M_{klij},$$

where

$i, j, k = 1, 2, 3,$

X_i volume density of the body forces,
Y_i volume density of the body couples,
τ_{ij} asymmetrical stress tensor,
m_{ij} couple stress tensor,
u_i displacement vector,
φ_i micro-rotation vector,
γ_{ij} strain tensor,
$\varkappa_{ij}$ curvature–twist tensor.

All the above quantities are functions of $x \in \Omega \subset E^3$, Ω posseses the Lipschitz boundary $\partial\Omega$. We assume that E_{ijkl}, K_{ijkl}, $M_{ijkl} \in L^\infty(\overline{\Omega})$, are bounded and measurable functions of x defined on $\overline{\Omega} = \Omega\cup\partial\Omega$. For the sake of simplicity in this paper we always deal with homogeneous boundary conditions:

(2.8) $$u_i = 0, \quad \varphi_i = 0 \quad \text{on} \quad \partial\Omega.$$

Besides we assume that there exists $c > 0$ such that for each $x \in \overline{\Omega}$

(2.9) $$E_{ijkl}\gamma_{ij}\gamma_{kl}+2K_{ijkl}\gamma_{ij}\varkappa_{kl}+M_{ijkl}\varkappa_{ij}\varkappa_{kl} \geqslant c\sum_{i,j=1}^{3}(\gamma_{ij}^2+\varkappa_{ij}^2)$$

which means that energy of deformation per unit volume is uniformly positive definite.

We denote

(2.10) $$\{u_1, u_2, u_3, \varphi_1, \varphi_2, \varphi_3\} \equiv \{u_i, \varphi_i\} \equiv u,$$

(2.11) $$\{v_1, v_2, v_3, \psi_1, \psi_2, \psi_3\} \equiv \{v_i, \psi_i\} \equiv v$$

and assume that

$$u, v \in V = (H_0^1(\Omega))^6.$$

The bilinear form $a(u, v)$ on $V \times V$ is defined by

$$a(v, u) = \int_{\Omega} [E_{ijkl}\gamma_{ij}(v)\gamma_{kl}(u) + K_{ijkl}\{\gamma_{ij}(u)\varkappa_{kl}(v) + \gamma_{ij}(v)\varkappa_{kl}(u)\} + M_{ijkl}\varkappa_{ij}(v)\varkappa_{kl}(u)]dx, \tag{2.12}$$

where

$$\begin{aligned} \gamma_{ij}(u) &= u_{j,i} - \varepsilon_{ijk}\varphi_k, \quad & \varkappa_{ij}(u) &= \varphi_{j,i}, \\ \gamma_{ij}(v) &= v_{j,i} - \varepsilon_{ijk}\psi_k, \quad & \varkappa_{ij}(v) &= \psi_{j,i}. \end{aligned} \tag{2.13}$$

Obviously $a(u, v) = a(v, u)$.

The weak formulation of the boundary value problem (2.1)—(2.8) reads

$$\begin{cases} \text{find}\, u \in V, \\ \forall v \in V \quad a(u, v) = F(v), \end{cases} \tag{2.14}$$

where

$$\begin{aligned} F(v) &= \int_{\Omega} (X_i v_i + Y_i \psi_i)dx = \int_{\Omega} fv\,dx, \quad X_i, Y_i \in L^2(\Omega), \\ f &= \{X_i, Y_i\}. \end{aligned} \tag{2.15}$$

We note that Cosserat continuum with different boundary conditions can be studied by means of the same variatinal procedure provided that a space V is properly chosen such that

$$(H_0^1(\Omega))^6 \subset V \subset (H^1(\Omega))^6.$$

The existence and uniqueness of the solution of the problem (2.14) was proved by I. HLAVÁCEK and M. HLAVÁCEK [6].

3. Homogenization

3.1. Formulation of the problem

We assume that Cosserat medium has a periodic structure defined as follows:

$$\text{let} \quad Y = [0, Y_1] \times [0, Y_2] \times [0, Y_3] \subset R^3.$$

After [1] we shall call it a basic cell.

Moreover, we assume that the functions

$$E_{ijkl}(y), K_{ijkl}(y), M_{ijkl}(y) \in L^\infty(Y) \tag{3.1}$$

and can be extended to the whole R^3 as Y — periodic functions. Next we define εY — periodic coefficients by

$$\begin{aligned} E^\varepsilon_{ijkl}(x) &= E_{ijkl}\left(\frac{x}{\varepsilon}\right), \\ K^\varepsilon_{ijkl}(x) &= K_{ijkl}\left(\frac{x}{\varepsilon}\right), \\ M^\varepsilon_{ijkl}(x) &= M_{ijkl}\left(\frac{x}{\varepsilon}\right), \quad \text{where} \quad y = \frac{x}{\varepsilon}. \end{aligned} \tag{3.2}$$

For a fixed $\varepsilon > 0$ we formulate the following problem:

$$(3.3)\qquad \begin{cases} \text{find}\, u^\varepsilon \in V, \text{ such that} \\ \forall v \in V,\ a^\varepsilon(u^\varepsilon, v) = F(v), \end{cases}$$

where

$$(3.4)\qquad a^\varepsilon(u^\varepsilon, v) = \int_\Omega [E^\varepsilon_{ijkl}(x)\gamma_{ij}(v)\varkappa_{kl}(u^\varepsilon) + K^\varepsilon_{ijkl}(x)\{\gamma_{ij}(v)\varkappa_{kl}(u^\varepsilon) + \gamma_{ij}(u^\varepsilon)\varkappa_{kl}(v)\} + M^\varepsilon_{ijkl}(x)\varkappa_{ij}(v)\varkappa_{kl}(u^\varepsilon)]dx.$$

The results presented in the preceding section imply that for a fixed $\varepsilon > 0$ a unique solution $u^\varepsilon \in V$ of the problem (3.3) exists.

The problem of homogenization consists in investigation the limit of u^ε when ε tends to zero.

3.2. Homogenization procedure

To study the behaviour of u^ε when ε tends to zero we use the energy method of homogenization. This method was originally proposed by L. TARTAR [2], see also Ref. [1, 3, 9].

We introduce the space

$$(3.5)\qquad W(Y) = \{v | v \in (H^1(Y))^6,\quad v \text{ takes equal values on opposite faces of } Y\},$$

and the bilinear form defined on $W(Y) \times W(Y)$

$$(3.6)\qquad a_Y(u, v) = \int_Y [E_{ijkl}(y) u_{j,i} v_{l,k} + K_{ijkl}(y)\{v_{j,i}\varphi_{l,k} + u_{j,i}\psi_{l,k}\} + M_{ijkl}(y)\varphi_{j,i}\psi_{l,k}]dy.$$

It is clear that $a_Y(u, v) = a_Y(v, u)$.

We introduce the vectors χ^{kL} which are solution of the problem

$$(3.7)\qquad \begin{cases} \chi^{kL} \in W(Y) \\ a^*_Y(\chi - P, w) = 0, \quad \forall w \in W(Y), \end{cases}$$

where the bilinear form a^*_Y is the adjont of a_Y and

$$(3.8)\qquad (P^{kL})_M = y_k \delta_{LM}, \quad k = 1, 2, 3, \quad L, M = 1, \ldots, 6.$$

The variational problem (3.7) on the cell Y has a unique solution (up to an additive constant) [3].

We shall now formulate and prove

THEOREM 1. *If $E_{ijkl}(y)$, $K_{ijkl}(y)$, $M_{ijkl}(y)$ are Y—periodic on R^3, strictly positive and satisfy the functions* (3.1), *and if the forces X_i and couples Y_i are elements of $L^2(\Omega)$, then the solution u^ε of the problem* (3.3) *converges weakly in the space V to u*

$$(3.9)\qquad u^\varepsilon \to u \text{ in } V \text{ weakly}$$

where u is a unique solution of the problem

$$(3.10)\qquad \begin{cases} u \in V \\ \mathscr{A}(u, v) = F(v), \quad \forall v \in V. \end{cases}$$

The bilinear form $\mathscr{A}(u,v)$ *is defined as*

$$\mathscr{A}(u,v) = \int_\Omega [\bar{E}_{ijkl}\gamma_{ij}(u)\varkappa_{kl}(v) + \bar{K}_{ijkl}\{\gamma_{ij}(v)\varkappa_{kl}(u) + \gamma_{ij}(u)\varkappa_{kl}(v)\} + \bar{M}_{ijkl}\varkappa_{ij}(u)\varkappa_{kl}(v)]dx, \tag{3.11}$$

where

$$\begin{aligned}\bar{E}_{ijkl} &= \frac{1}{|Y|}a_Y^*(\chi^{ij}-P^{ij},\ \chi^{kl}-P^{kl}),\\ \bar{K}_{ijkl} &= \frac{1}{|Y|}a_Y^*(\chi^{ij}-P^{ij},\chi^{k(l+3)}-P^{k(l+3)}),\\ \bar{M}_{ijkl} &= \frac{1}{|Y|}a_Y^*(\chi^{i(j+3)}-P^{i(j+3)},\chi^{k(l+3)}-P^{k(l+3)}).\end{aligned} \tag{3.12}$$

Proof. For the simplicity of notation we introduce the block matrix

$$A_{iJkL} = \begin{bmatrix} E_{ijkl} & \text{for} & J=j, & L=l,\\ K_{ijkl} & \text{for} & J=j, & L=l+3,\\ K_{klij} & \text{for} & J=j+3, & L=l,\\ M_{ijkl} & \text{for} & J=j+3, & L=l+3, \end{bmatrix} \tag{3.13}$$

where $i,j,k,l = 1,2,3$; $J, L = 1, \ldots, 6$, similarly $\bar{A}_{iJkL}$, A^ε_{iJkL}, and a vector

$$\Gamma_{kL} = \begin{bmatrix} \gamma_{kl}, & \text{if } L=l\\ \varkappa_{kl}, & \text{if } L=l+3\end{bmatrix} \tag{3.14}$$

besides

$$\Gamma_{kL}(u) = \partial_k(u_L) - B_k(u_L) = (\nabla u - Bu)_{kL}, \tag{3.15}$$

where

$$u_L = \begin{bmatrix} u_l, & \text{if} & L=l\\ \varphi_l, & \text{if} & L=l+3\end{bmatrix}, \quad B_k(u_L) = \begin{bmatrix} \varepsilon_{klm}\varphi_m, & \text{if} & L=l\\ 0\quad, & \text{if} & L=l+3\end{bmatrix}.$$

Then the bilinear forms (2.12), (3.4), (3.6) and (3.11) may be written respectively in the form

$$a(u,v) = \int_\Omega \Gamma^T(u)A\Gamma(v)\,dx,$$

$$a^\varepsilon(u^\varepsilon,v) = \int_\Omega \Gamma^T(u^\varepsilon)A^\varepsilon\Gamma(v)\,dx,$$

$$a_Y(u,v) = \int_Y (\nabla u)^T A(\nabla v)\,dy,$$

$$\mathscr{A}(u,v) = \int_\Omega \Gamma^T(u)\bar{A}\Gamma(v)\,dx.$$

The proof will be carried out in two parts.

Part 1. Taking $v = u^\varepsilon$ in the problem (3.3) we have from the existence results for the problem (3.3) the following a priori estimate:

$$\|u^\varepsilon\|_V \leqslant C, \quad \forall \varepsilon > 0. \tag{3.16}$$

We denote

$$T^{\varepsilon}_{iJ} = \begin{bmatrix} \tau^{\varepsilon}_{ij}, & \text{if} & J = j \\ m^{\varepsilon}_{ij}, & \text{if} & J = j+3 \end{bmatrix}, \tag{3.17}$$

where

$$\tau^{\varepsilon}_{ij} = E^{\varepsilon}_{ijkl}\gamma_{kl}(u^{\varepsilon}) + K^{\varepsilon}_{ijkl}\varkappa_{kl}(u^{\varepsilon}),$$
$$m^{\varepsilon}_{ij} = K^{\varepsilon}_{klij}\gamma_{kl}(u^{\varepsilon}) + M^{\varepsilon}_{ijkl}\varkappa_{kl}(u^{\varepsilon}).$$

According to the notation (3.13) and (3.14),

$$T^{\varepsilon}_{iJ} = A^{\varepsilon}_{iJkL}\Gamma_{kL}(u^{\varepsilon}). \tag{3.18}$$

From the assumption (3.1) and estimate (3.16) we have

$$|T^{\varepsilon}_{iJ}|_{L^2(\Omega)} \leqslant C, \quad \forall \varepsilon > 0. \tag{3.19}$$

Therefore we can extract subsequences from the sequences u^{ε} and T^{ε}_{iJ}, still denoted by u^{ε} and T^{ε}_{iJ} such that

$$\begin{cases} u^{\varepsilon} \rightharpoonup u & \text{in } V \text{ weakly}, \\ T^{\varepsilon}_{iJ} \rightharpoonup T_{iJ} & \text{in } L^2(\Omega) \text{ weakly}. \end{cases} \tag{3.20}$$

Passing to the limit in

$$\int_{\Omega} T^{\varepsilon}_{iJ}\Gamma_{iJ}(v)\,dx = F(v), \quad \forall v \in V, \tag{3.21}$$

we obtain

$$\int_{\Omega} T_{iJ}\Gamma_{iJ}(v)\,dx = F(v), \quad \forall v \in V. \tag{3.22}$$

Part 2. Let $P(y)$ be the vector field on Y the components of which are a homogeneous polynomial of degree 1 in y. Then there exists a unique solution w up to an additive constant vector of the following problem:

$$\begin{cases} a^{*}_{Y}(w, v) = 0, & \forall v \in W(Y), \\ w - P \in W(Y). \end{cases} \tag{3.23}$$

Next we define

$$w^{\varepsilon}(x) = \varepsilon w\left(\frac{x}{\varepsilon}\right) = P(x) + \varepsilon\pi\left(\frac{x}{\varepsilon}\right) \tag{3.24}$$

where

$$\pi(y) \in W(Y).$$

The vector field $\pi\left(\dfrac{x}{\varepsilon}\right)$ is bounded in $(L^2(\Omega))^6$.

It follows from the relation (3.24) that

$$w^{\varepsilon} \to P \quad \text{in } (L^2(\Omega))^6 \text{ strongly}. \tag{3.25}$$

Besides $w^{\varepsilon}(x)$ satisfies

$$\frac{\partial}{\partial x_i}\left(A^{\varepsilon}_{kLiJ}\frac{\partial}{\partial x_k}w^{\varepsilon}_L\right) = 0 \quad \text{in } \Omega. \tag{3.26}$$

We infer from Eq. (3.26) that w^ε satisfies

$$\int_\Omega \left(A^\varepsilon_{kLiJ} \frac{\partial}{\partial x_k} w^\varepsilon_L \frac{\partial}{\partial x_i} v_J \right) dx = 0, \quad \forall v \in (H^1_0(\Omega))^6. \tag{3.27}$$

Denote by $\mathscr{D}(\Omega)$ the space of infinitely differentiable functions φ with compact supports in Ω.

Taking $v = \varphi w^\varepsilon$, $\varphi \in \mathscr{D}(\Omega)$ in the problem (3.3) and $v = \varphi u^\varepsilon$ in Eq. (3.27) and then subtracting, we obtain

$$a^\varepsilon(u^\varepsilon, \varphi w^\varepsilon) - \int_\Omega \left[A^\varepsilon_{kLiJ} \frac{\partial}{\partial x_k} w^\varepsilon_L \frac{\partial}{\partial x_i} (\varphi u^\varepsilon_J) \right] dx = F(\varphi w^\varepsilon). \tag{3.28}$$

After transformations we have

$$\int_\Omega [T^\varepsilon_{iJ}(\partial_i \varphi) w^\varepsilon_J - A^\varepsilon_{kLiJ} \partial_k w^\varepsilon_L (\partial_i \varphi) u^\varepsilon_J - T^\varepsilon_{iJ} \varphi B_i(w^\varepsilon_J) - A^\varepsilon_{kLiJ} \partial_k w^\varepsilon_L \varphi B_i(u^\varepsilon_J)] dx = \int_\Omega f_L \varphi w^\varepsilon_L dx. \tag{3.29}$$

. We can go to the limit in Eq. (3.29): w^ε_L converges strongly in $L^2(\Omega)$ to P_L and T^ε_{iJ} converges weakly in $L^2(\Omega)$ to T_{iJ}; besides

$$A^\varepsilon_{kLiJ} \partial_k w^\varepsilon_L = A_{kLiJ}(y) \partial_k w_L(y)|_{y=\frac{x}{\varepsilon}}$$

is εY periodic and converges weakly in $L^2(\Omega)$ to an average value on Y of the value $A_{kLiJ}(y)\partial_k w_L(y)$. We denote it by $M_{iJ}(P)$. At last u^ε_J converges strongly in $L^2(\Omega)$ to u_J.

So in the limit we get

$$\int_\Omega [T_{iJ}(\partial_i \varphi) P_J - M_{iJ}(P)(\partial_i \varphi) u_J - T_{iJ} \varphi B_i(P_J) - M_{iJ}(P) \varphi B_i(u_J)] dx = \int_\Omega f_L \varphi P_L dx. \tag{3.30}$$

Using Eq. (3.22) to eliminate f_L from Eq. (3.30), after transformations we obtain

$$\int_\Omega \varphi [M_{iJ}(P) \Gamma_{iJ}(u) - T_{iJ} \partial_i P_J] dx = 0, \quad \forall \varphi \in \mathscr{D}(\Omega). \tag{3.31}$$

So we get

$$M_{iJ}(P) \Gamma_{iJ}(u) = T_{iJ} \partial_i P_J. \tag{3.32}$$

Choosing $P = P^{rS}$ where

$$(P^{rS})_J = y_r \delta_{SJ}, \tag{3.33}$$

we obtain

$$T_{rS} = M_{iJ}(P^{rS}) \Gamma_{iJ}(u), \tag{3.34}$$

where

$$\bar{A}_{rSiJ} = M_{iJ}(P^{rS}) = \frac{1}{|Y|} \int_Y A_{kLiJ}(y) \partial_k w^{rS}_L(y) dy. \tag{3.35}$$

The vector w^{rS} is associated with P^{rS} by the problem (3.23).

Taking

$$-\chi^{rS} = w^{rS} - P^{rS} \tag{3.36}$$

what follows that χ^{rS} is defined by

$$\begin{cases} a_Y^*(\chi^{rS} - P^{rS}, w) = 0, & \forall w \in W(Y), \\ \chi^{rS} \in W(Y) \end{cases} \tag{3.37}$$

we can rewrite the relation (3.35) in the form

$$\bar{A}_{rSiJ} = \frac{1}{|Y|} \int_Y A_{kLmN}\, \partial_k (P_L^{rS} - \chi_L^{rS})\, \partial_m (P_N^{iJ})\, dy \tag{3.38}$$

which is equivalent to the relations (3.12).

The uniqueness of the solution of the problem (3.10) is due to L. TARTAR [2].

REMARK 1. From the above results it follows that, in the limit, inhomogeneous Cosserat medium behaves as homogeneous Cosserat medium with the constitutive relations

$$\begin{cases} \tau_{ij} = \bar{E}_{ijkl}\gamma_{kl} + \bar{K}_{ijkl}\varkappa_{kl}, \\ m_{ij} = \bar{K}_{klij}\gamma_{kl} + \bar{M}_{ijkl}\varkappa_{kl}. \end{cases} \tag{3.39}$$

Then the constitutive law (3.39) is a homogenized constitutive law which is evidently independent of boundary conditions.

4. Correctors

As we have seen in the previous section, $u^\varepsilon - u$ converges to zero in V only weakly and not strongly. We can, however, define a corrector Θ^ε such that

$$u^\varepsilon - u - \Theta^\varepsilon \to 0 \quad \text{in } V \text{ strongly}. \tag{4.1}$$

For this purpose we first introduce cut-off functions m_ε having the following properties [1]:

$$m_\varepsilon \in \mathcal{D}(\Omega),$$

$$m_\varepsilon = \begin{cases} 0, & \text{if} \quad \operatorname{dist}(x, \partial\Omega) \leqslant \varepsilon, \\ 1, & \text{if} \quad \operatorname{dist}(x, \partial\Omega) \geqslant 2\varepsilon, \end{cases} \tag{4.2}$$

$\forall \beta$, $\varepsilon^\beta |m_\varepsilon^{(\beta)}(x)| \leqslant C_\beta$, where C_β depends on β but does not depend on ε.

We then define the corrector Θ^ε as follows:

$$\Theta_L^\varepsilon = -\varepsilon m_\varepsilon \chi_L^{kM}\left(\frac{x}{\varepsilon}\right)\Gamma_{kM}(u). \tag{4.3}$$

We can now formulate

THEOREM 2. *Assume that in addition to the hypotheses of Theorem 1 the following assumptions hold*:

$$\chi^{kM} \in (W^{1,\infty}(Y))^6, \quad u \in (H^2(\Omega))^6. \tag{4.4}$$

Then

$$z^{\varepsilon} = u^{\varepsilon} - u - \Theta^{\varepsilon} \to 0 \quad \text{in } V \text{ strongly}, \tag{4.5}$$

where Θ^{ε} *is defined by Eq.* (4.3).

The proof of Theorem 2 is omitted because it is similar to the one given in [1, 3, 5].

5. Example: one-dimensional case

For the case of isotropic material we have

$$K_{ijkl} = 0. \tag{5.1}$$

We assumed additionally that $E_{ijkl}(y)$, $M_{ijkl}(y)$ are y_3 — periodic functions only. They are constant as functions of y_1 and y_2. Then we have

$$\begin{cases} E_{ijkl} = E_1(y_3)\,\delta_{ik}\,\delta_{jl} + E_2(y_3)\,\delta_{il}\,\delta_{jk} + E_3(y_3)\,\delta_{ij}\,\delta_{kl}, \\ M_{ijkl} = M_1(y_3)\,\delta_{ik}\,\delta_{jl} + M_2(y_3)\,\delta_{il}\,\delta_{jk} + M_3(y_3)\,\delta_{ij}\,\delta_{kl}. \end{cases} \tag{5.2}$$

In this case we can calculate explicitly the form of homogenized coefficients.

By virtue of the relation (5.1) the „problem of the cell" has the following simple form:

$$\frac{\partial}{\partial y_i}\left(E_{ijkl}\frac{\partial}{\partial y_k}\chi_l^{mn}\right) = \frac{\partial}{\partial y_i}E_{ijmn}, \tag{5.3}$$

$$\frac{\partial}{\partial y_i}\left(M_{ijkl}\frac{\partial}{\partial y_k}\tilde{\chi}_l^{mn}\right) = \frac{\partial}{\partial y_i}M_{ijmn}. \tag{5.4}$$

Assuming that $\chi_l^{mn}(y_3)$, $\tilde{\chi}_l^{mn}(y_3)$ and integrating Eq. (5.3), we have

$$\frac{d}{dy_3}\chi_l^{mn} = E_1(E_{3j3l})^{-1}\delta_{3m}\,\delta_{jn} + E_2(E_{3j3l})^{-1}\delta_{3n}\,\delta_{jm} + E_3(E_{3j3l})^{-1}\delta_{3j}\,\delta_{mn} + (E_{3j3l})^{-1}c_{jnm}. \tag{5.5}$$

The constant of integration c_{jnm} can be calculated from the condition of the existence of the solution of Eq. (5.5) for χ_l^{mn} periodic in Y [1].

We have

$$c_{jnm} = -[(I_0^{-1}\delta_{j1}\,\delta_{n1} + I_0^{-1}\delta_{j2}\,\delta_{n2} + I_1^{-1}I_2\,\delta_{j3}\,\delta_{n3})\,\delta_{3m} + (I_0^{-1}I_4\,\delta_{j1}\,\delta_{m1} + I_0^{-1}I_4\,\delta_{j2}\,\delta_{m2} + I_1^{-1}I_5\,\delta_{j3}\,\delta_{m3})\,\delta_{3n} + I_1^{-1}I_7\,\delta_{j3}\,\delta_{mn}], \tag{5.6}$$

here

$$I_0 = \frac{1}{Y_3}\int_0^{Y_3}\frac{1}{E_1}dy_3, \qquad I_1 = \frac{1}{Y_3}\int_0^{Y_3}\frac{1}{E_1+E_2+E_3}dy_3,$$

$$I_2 = \frac{1}{Y_3}\int_0^{Y_3}\frac{E_1}{E_1+E_2+E_3}dy_3, \qquad I_4 = \frac{1}{Y_3}\int_0^{Y_3}\frac{E_2}{E_1}dy_3,$$

$$I_5 = \frac{1}{Y_3}\int_0^{Y_3}\frac{E_2}{E_1+E_2+E_3}dy_3, \qquad I_7 = \frac{1}{Y_3}\int_0^{Y_3}\frac{E_3}{E_1+E_2+E_3}dy_3.$$

The homogenized coefficients can be calculated from

$$\bar{E}_{ijkl} = \frac{1}{Y_3}\int_0^{Y_3}\left[E_{ijkl}(y_3) - E_{ij3h}\frac{d}{dy_3}\chi_h^{kl}\right]dy_3. \tag{5.7}$$

Using Eqs. (5.2) and (5.5) we have

$$\begin{aligned}\bar{E}_{ijkl} = {} & \bar{E}_1\,\delta_{ik}\delta_{jl} + \bar{E}_2\,\delta_{il}\,\delta_{jk} + (\bar{E}_3 - I_{13} + I_1^{-1}I_7^2)\,\delta_{ij}\,\delta_{kl}\\ & -(\bar{E}_1 - I_0^{-1})\,\delta_{i3}\,\delta_{j1}\,\delta_{3k}\,\delta_{l1} - (\bar{E}_1 - I_0^{-1})\,\delta_{i3}\,\delta_{j2}\,\delta_{l2}\,\delta_{3k}\\ & -(I_3 + I_8 - I_1^{-1}I_2^2 - I_2I_1^{-1}I_5)\,\delta_{i3}\,\delta_{j3}\,\delta_{l3}\,\delta_{3k}\\ & -(\bar{E}_2 - I_0^{-1})\,\delta_{i3}\,\delta_{j1}\,\delta_{k1}\,\delta_{3l} - (\bar{E}_2 - I_0^{-1}I_4)\,\delta_{i3}\,\delta_{j2}\,\delta_{k2}\,\delta_{3l}\\ & -(I_9 - I_2I_1^{-1}I_7 + I_{12} - I_5I_1^{-1}I_7)\,\delta_{i3}\,\delta_{j3}\,\delta_{kl}\\ & -(\bar{E}_2 - I_4I_0^{-1})\,\delta_{i1}\,\delta_{j3}\,\delta_{l1}\,\delta_{3k} - (\bar{E}_2 - I_4I_0^{-1})\,\delta_{i2}\,\delta_{j3}\,\delta_{l2}\,\delta_{3k}\\ & -(I_8 + I_{10} - I_1^{-1}I_2I_5 - I_1^{-1}I_5^2)\,\delta_{i3}\,\delta_{j3}\,\delta_{l3}\,\delta_{3k}\\ & -(I_{11} - I_4^2I_0^{-1})\,\delta_{i1}\,\delta_{j3}\,\delta_{k1}\,\delta_{l3} - (I_{11} - I_0^{-1}I_4^2)\,\delta_{i2}\,\delta_{j3}\,\delta_{k2}\,\delta_{l3}\\ & -(I_9 + I_{12} - I_7I_1^{-1}I_2 - I_7I_1^{-1}I_5)\,\delta_{ij}\,\delta_{l3}\,\delta_{3k},\end{aligned} \tag{5.8}$$

where

$$I_3 = \frac{1}{Y_3}\int_0^{Y_3}\frac{E_1^2}{E_1+E_2+E_3}\,dy_3, \qquad I_8 = \frac{1}{Y_3}\int_0^{Y_3}\frac{E_1E_2}{E_1+E_2+E_3}\,dy_3,$$

$$I_9 = \frac{1}{Y_3}\int_0^{Y_3}\frac{E_1E_3}{E_1+E_2+E_3}\,dy_3, \qquad I_{10} = \frac{1}{Y_3}\int_0^{Y_3}\frac{E_2^2}{E_1+E_2+E_3}\,dy_3,$$

$$I_{11} = \frac{1}{Y_3}\int_0^{Y_3}\frac{E_2^2}{E_1}\,dy_3, \qquad I_{12} = \frac{1}{Y_3}\int_0^{Y_3}\frac{E_2E_3}{E_1+E_2+E_3}\,dy_3,$$

$$I_{13} = \frac{1}{Y_3}\int_0^{Y_3}\frac{E_3^2}{E_1+E_2+E_3}\,dy_3, \qquad \bar{E}_i = \frac{1}{Y_3}\int_0^{Y_3}E_i\,dy_3, \quad i = 1, 2, 3.$$

From Eq. (5.8) we have

$$\begin{aligned}\bar{E}_{1111} &= \bar{E}_{2222} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3 - I_{13}I_1^{-1} + I_7^2,\\ \bar{E}_{1122} &= \bar{E}_{2211} = \bar{E}_3 - I_{13} + I_1^{-1}I_7^2,\\ \bar{E}_{2233} &= \bar{E}_{3322} = \bar{E}_{1133} = \bar{E}_{3311} = \bar{E}_3 - I_9 + I_2I_1^{-1}I_7 - I_{12} + I_5I_1^{-1}I_7 - I_{13} + I_1^{-1}I_7^2,\\ \bar{E}_{3333} &= \bar{E}_1 + \bar{E}_2 + \bar{E}_3 - I_3 - 2I_8 + I_1^{-1}I_2^2 + 2I_2I_1^{-1}I_5 - 2I_9\\ &\qquad + 2I_2I_1^{-1}I_7 - I_{10} + I_1^{-1}I_5^2 - 2I_{12} + 2I_5I_1^{-1}I_7 - I_{13} + I_1^{-1}I_7^2,\\ \bar{E}_{1212} &= \bar{E}_{2121} = \bar{E}_1,\\ \bar{E}_{1313} &= \bar{E}_{2323} = \bar{E}_1 - I_{11} + I_0^{-1}I_4^2,\\ \bar{E}_{3131} &= \bar{E}_{3232} = I_0^{-1}.\end{aligned}$$

In the same way we can calculate $\bar{M}_{ijkl}$.

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