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Some theorems in generalized micropolar thermoelasticity

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A THEOREM on the uniqueness of solution, a generalized Hamilton's principle and a reciprocal theorem for dynamical mixed boundary value problems are obtained in the context of a linear anisotropic micropolar thermoelasticity theory, which predicts a finite speed of propagation of thermo-mechanical signals.

W ramach liniowej termosprężystości anizotropowych ciał mikropolarnych wyprowadzono twierdzenie o jednoznaczności rozwiązań, uogólnioną zasadę Hamiltona oraz twierdzenie o wzajemności dla mieszanych, dynamicznych zagadnień brzegowych; uwzględniono przypadek skończonej prędkości propagacji zaburzeń termomechanicznych.

В рамках линейной термоупругости анизотропных микрополярных тел выведена теорема единственности решений, обобщенный принцип Гамильтона и теорема взаимности для смешанных, динамических краевых задач; учтен случай конечной скорости распространения термомеханических возмущений.

1. Introduction

THE THERMOELASTICITY theory formulated by GREEN and LINDSAY [1] and by SUHUBI [2] has aroused much interest in recent years. Unlike the conventional coupled thermoelasticity theory [3], this theory includes the temperature-rate among the constitutive variables and consequently predicts a finite speed for the propagation of thermo-mechanical disturbances. Since pure thermal signals propagating with a finite speed have actually been observed in solids [4, 5], this theory, like some other generalized thermoelasticity theories (e.g. [6, 7]), is physically more realistic than the coupled theory, and problems revealing interesting phenomena characterizing the theory are contained in [8—17]. BOSCHI and IESAN [18], and DOST and TABARROK [19] have extended this theory to micropolar elastic materials, and CHANDRASEKHARAIAH [20] has formulated an analogous theory for piezoelectric materials.

The object of this paper is to prove three main theorems, viz., (i) a uniqueness theorem, (ii) a variational principle of the Hamilton-type, and (iii) a reciprocal theorem of the Betti-Rayleigh type, for linearized anisotropic micropolar thermoelastic interactions, by employing the equations obtained in [18, 19]. In Sect. 2, we summarize the governing equations of the generalized (linear) micropolar thermoelasticity and formulate an initial, mixed boundary value problem. In Sect. 3, we obtain the equation of energy balance in terms of the Biot's potential [21], and in Sect. 4, we employ it to establish a uniqueness theorem. In Sect. 5, we establish a Hamilton-type variational principle in terms of a single functional and in Sect. 6, we deduce a Betti-Rayleigh type reciprocal theorem. Throughout the paper, we show how some known results follow as limiting cases of those obtained here.

2. Basic equations

In the context of the theory formulated in [18, 19], the governing equations of linear micropolar thermoelastic interactions in a homogeneous and anisotropic solid are:

(i) **Kinematic equations**

$$e_{ij} = u_{j,i} + \varepsilon_{jik}\phi_k. \tag{2.1}$$

(ii) **Equations of motion**

$$t_{ij,j} + \varrho F_i = \varrho \ddot{u}_i, \tag{2.2}$$

$$m_{ji,j} + \varepsilon_{ijk} t_{jk} + \varrho M_i = \varrho J_{ij} \ddot{\phi}_j. \tag{2.3}$$

(iii) **Equation of entropy**

$$\varrho\theta_0 \dot{S} + q_{k,k} = \varrho r. \tag{2.4}$$

(iv) **Constitutive equations**

$$t_{ij} = A_{ijkl} e_{kl} + B_{ijkl}\phi_{l,k} + a_{ij}(\theta + \alpha\dot{\theta}), \tag{2.5}$$

$$m_{ij} = B_{klij} e_{kl} + C_{ijkl}\phi_{l,k} + c_{ij}(\theta + \alpha\dot{\theta}), \tag{2.6}$$

$$q_i = -\theta_0(b_i\dot{\theta} + k_{ij}\theta_{,j}), \tag{2.7}$$

$$\varrho S = a + d\theta + h\dot{\theta} - b_i\theta_{,i} - a_{ij}e_{ij} - c_{ij}\phi_{j,i}. \tag{2.8}$$

Unless stated to the contrary, the notation and the symbols in these equations and those to follow are as explained in [18].

Eliminating t_{ij} from Eqs. (2.2) and (2.5) and m_{ij} from Eqs. (2.3) and (2.6), and using Eq. (2.1), we obtain

$$A_{ijkl}(u_{l,kj} + \varepsilon_{lkm}\phi_{m,j}) + B_{ijkl}\phi_{l,kj} + a_{ij}(\theta + \alpha\dot{\theta})_{,j} + \varrho F_i = \varrho\ddot{u}_i, \tag{2.9}$$

$$B_{klji}(u_{l,kj} + \varepsilon_{lkm}\phi_{m,j}) + C_{jikl}\phi_{l,kj} + c_{ji}(\theta + \alpha\dot{\theta})_{,j} + \varepsilon_{ijk}\{A_{jkrs}(u_{s,r} + \varepsilon_{srm}\phi_m) + B_{jkrs}\phi_{s,r} + a_{jk}(\theta + \alpha\dot{\theta})\} + \varrho M_i = \varrho J_{ij}\ddot{\phi}_j. \tag{2.10}$$

Eliminating q_i and S from Eqs. (2.4), (2.7) and (2.8) and using Eq. (2.1), we obtain

$$(\varrho r/\theta_0) + k_{ij}\theta_{,ji} - d\dot{\theta} - h\ddot{\theta} + 2b_i\dot{\theta}_{,i} + a_{ij}(\dot{u}_{j,i} + \varepsilon_{jik}\dot{\phi}_k) + c_{ij}\dot{\phi}_{j,i} = 0. \tag{2.11}$$

Evidently, Eqs. (2.9) and (2.10) are the equations of motion and (2.11) is the equation of thermo-mechanical transport, expressed in terms of the displacement components u_i, microrotation components ϕ_i and the temperature change θ. All these equations are coupled together and form a complete system of field equations of the theory considered. Apart from the notation, these equations are identical with Eqs. (4.14) of [19].

It has been shown [18, 19] that the material constants satisfy the symmetry conditions

$$A_{ijkl} = A_{klij}, \quad C_{ijkl} = C_{klij}, \quad k_{ij} = k_{ji} \tag{2.12}$$

and the inequality

$$(d\alpha - h)\dot{\theta}^2 + 2b_i\theta_{,i}\dot{\theta} + k_{ij}\theta_{,i}\theta_{,j} \geqslant 0. \tag{2.13}$$

Since $\theta_{,i}$ and $\dot{\theta}$ are arbitrary real variables, it readily follows that

$$d\alpha - h \geqslant 0, \quad k_{ij}y_iy_j \geqslant 0 \tag{2.14}$$

for arbitrary real variables y_i.

We recognize, from Eq. (2.11), that d is the specific heat capacity and k_{ij} is the thermal conductivity tensor. From the inequalities $(2.12)_3$ and $(2.14)_2$, it follows that k_{ij} is a positive definite symmetric tensor—a result well known in the coupled thermoelasticity theory [3]. Because of its physical meaning, we may take

$$d > 0. \tag{2.15}$$

If we set $h = \alpha_0 d$ and postulate that

$$\alpha_0 \geqslant 0, \tag{2.16}$$

then the inequality $(2.14)_1$ yields

$$\alpha \geqslant \alpha_0 \geqslant 0. \tag{2.17}$$

It is evident that if $\alpha = 0$, then $\alpha_0 = 0$ and that if $\alpha_0 > 0$, then $\alpha > 0$. We also see that if $\alpha_0 > 0$, then the thermo-mechanical transport equation (2.11) is of the hyperbolic-type implying a finite speed of propagation of thermal disturbances (-second sound). Consequently, it follows that if t_{ij} and m_{ij} are independent of $\dot{\theta}$, then S is also independent of $\dot{\theta}$, see Eqs. (2.5), (2.6) and (2.8), and that if the theory admits second sound, then t_{ij}, m_{ij} and S depend on $\dot{\theta}$. In the limiting case when $\alpha = 0$ (and consequently $\alpha_0 = 0$) and $b_i = 0$, the theory reduces to the conventional micropolar thermoelasticity theory [22]. Indeed, if we set $\alpha = b_i = 0$, in the field equations (2.9)—(2.11) and specialize the equations for isotropic materials, we recover Eqs. (4.1), (4.2) and (4.4) of [23].

If we assume that at time $t = 0$, the body is at rest in its initial undeformed state and is at the reference temperature θ_0, zero temperature-rate and zero entropy, then the following initial conditions are to be satisfied:

$$\begin{aligned} u_i(x, 0) = \dot{u}_i(x, 0) = \phi_i(x, 0) = \dot{\phi}_i(x, 0) = \\ = \theta(x, 0) = \dot{\theta}(x, 0) = S(x, 0) = 0, \quad x \in B, \end{aligned} \tag{2.18}$$

with B being the initial configuration of the body.

Further, if we assume that for $t \geqslant 0$ (i) the surface forces are prescribed on a part ∂B_1 and the displacements on the remaining part ∂B_1^c of the boundary surface ∂B of B, (ii) the surface couples are prescribed on a part ∂B_2 and the microrotations on the remaining part ∂B_2^c of ∂B, and (iii) the heat flux is prescribed on a part ∂B_3 and the temperature on the remaining part ∂B of ∂B_3^c, then the following conditions are to be satisfied for $t \geqslant 0$:

$$t_{ij} n_j = T_i \quad \text{on } \partial B_1, \qquad u_i = U_i \quad \text{on } \partial B_1^c; \tag{2.19}$$

$$m_{ji} n_j = L_i \quad \text{on } \partial B_2, \qquad \phi_i = \Phi_i \quad \text{on } \partial B_2^c; \tag{2.20}$$

$$q_i n_i = \theta_0 Q \quad \text{on } \partial B_3, \qquad \theta = \Theta \quad \text{on } \partial B_3^c. \tag{2.21}$$

In these conditions, T_i, U_i, L_i, Φ_i, Θ and Q are the prescribed functions in their respective domains and n_i is the unit outward normal to ∂B. With $F_i(x, t)$, $M_i(x, t)$ and $r(x, t)$ as the prescribed functions for $x \in B$ and $t \geqslant 0$, the problem of determining the field variables $u_i(x, t)$, $\phi_i(x, t)$ and $\theta(x, t)$ for $x \in B$ and $t > 0$, by solving Eqs. (2.9)—(2.11) constitutes an initial mixed boundary value problem in the context of the theory considered. We will refer to this problem as the problem associated with the system:

$$\{(F_i, M_i, r); (T_i, U_i, L_i, \Phi_i, \Theta, Q); (u_i, \phi_i, \theta)\}.$$

3. Energy equation

We now obtain the equation of energy balance in terms of the generalized free energy function V introduced by BIOT [21] through the equation

$$V = U - S\theta_0 \tag{3.1}$$

where U is the internal energy.

The energy function considered in [18] is given by

$$\psi = U - S\phi \tag{3.2}$$

and has the following explicit form, under the initial conditions (2.18):

$$\varrho\psi = \sigma_0 - \frac{1}{2}d\theta^2 - d\alpha\theta\dot{\theta} - \frac{1}{2}h\alpha\dot{\theta}^2 + \alpha b_i\theta_{,i}\dot{\theta} + a_{ij}e_{ij}(\theta + \alpha\dot{\theta}) + c_{ij}\phi_{j,i}(\theta + \alpha\dot{\theta}) + \frac{1}{2}\alpha k_{ij}\theta_{,i}\theta_{,j} + \frac{1}{2}A_{ijkl}e_{ij}e_{kl} + \frac{1}{2}C_{ijkl}\phi_{j,i}\phi_{l,k} + B_{ijkl}e_{ij}\phi_{l,k}. \tag{3.3}$$

The scalar function ϕ appearing in (3.2) is also given by [18]

$$\phi = \theta_0 + \theta + \alpha\dot{\theta} + \beta\theta\dot{\theta} + \frac{1}{2}\gamma\dot{\theta}^2. \tag{3.4}$$

Eliminating U from Eqs. (3.1) and (3.2) and substituting for ψ, φ and S from Eqs. (3.3), (3.4) and (2.8), we obtain the following quadratic form expansion for the Biot's energy function V:

$$\varrho V = \frac{1}{2}(d\theta^2 + h\alpha\dot{\theta}^2 + \alpha k_{ij}\theta_{,i}\theta_{,j}) + h\theta\dot{\theta} - b_i\theta_{,i}\theta + \frac{1}{2}A_{ijkl}e_{ij}e_{kl} + \frac{1}{2}C_{ijkl}\phi_{j,i}\phi_{l,k} + B_{ijkl}e_{ij}\phi_{l,k}. \tag{3.5}$$

In obtaining this expression we have made use of the initial conditions (2.18) and have neglected third and higher degree terms in the field variables.

The kinetic energy per unit mass is given by [18]

$$T = \frac{1}{2}(\dot{u}_i\dot{u}_i + J_{ij}\dot{\phi}_i\dot{\phi}_j). \tag{3.6}$$

With the aid of Eqs. (2.5)—(2.7), (2.9)—(2.11), (3.5) and (3.6), and the divergence theorem, we obtain

$$\left[\frac{d}{dt}\int_m (T+V)dm\right] + N = \int_m \left\{F_i\dot{u}_i + M_i\dot{\phi}_i + \frac{r}{\theta_0}(\theta + \alpha\dot{\theta})\right\}dm + \int_{\partial B}\left\{t_{ij}\dot{u}_j + m_{ij}\dot{\phi}_j - \frac{q_i}{\theta_0}(\theta + \alpha\dot{\theta})\right\}n_i\,dA, \tag{3.7}$$

where

$$N = \int_B \{(d\alpha - h)\dot{\theta}^2 + 2b_i\theta_{,i}\dot{\theta} + k_{ij}\theta_{,i}\theta_{,j}\}dB \tag{3.8}$$

and m is the mass included in B.

The equation (3.7) is the desired equation of energy balance. This is a generalization to micropolar thermoelasticity of Eq. (20) of [24]. If we set $\alpha = b_i = 0$ in this equation, we recover Eq. (5.11) of [23], obtained for isotropic materials.

We note that because of the inequality (2.13), the integrand on the right-hand side of Eq. (3.8) is nonnegative, and accordingly we have

$$N \geqslant 0. \tag{3.9}$$

4. Uniqueness theorem

We now employ Eq. (3.7) to establish the following uniqueness theorem.

THEOREM. *If the Biot's energy function V is positive definite, then there exists at most one solution for the problem associated with the system*:

$$\{(F_i, M_i, r); (T_i, U_i, L_i, \Phi_i, \Theta, Q); (u_i, \phi_i, \theta)\}.$$

Proof. To establish the theorem it is sufficient to show that for $F_i = M_i = r \equiv 0$ in B and $T_i = U_i = \Theta = Q \equiv 0$ in their respective domains on ∂B, the solution is trivial. For $F_i = M_i = r \equiv 0$ and the homogeneous boundary conditions, Eq. (3.7) simplifies to

$$\frac{d}{dt}\int_m (T+V)\,dm = -N. \tag{4.1}$$

Since the right side of this equation is non-positive, because of (3.9), it follows that $\int_m (T+V)\,dm$ is a nonincreasing function of time. At $t = 0$, we have $T = V = 0$, in view of the initial conditions (2.8). Consequently, we should have

$$\int_m (T+V)\,dm \leqslant 0 \quad \text{for} \quad t \geqslant 0. \tag{4.2}$$

Since $T \geqslant 0$ by definition [18], it follows that

$$T = V = 0 \quad \text{in} \quad B \quad \text{for} \quad t \geqslant 0, \tag{4.3}$$

provided V is positive definite.

The equations (4.3) readily yield the trivial solution

$$u_i(x, t) = \phi_i(x, t) = \theta(x, t) \equiv 0 \quad \text{for} \quad x \in B, \quad t \geqslant 0. \tag{4.4}$$

This completes the proof.

It may be remarked that this uniqueness theorem is a generalization to micropolar thermoelasticity of the corresponding theorem proved in [24].

5. Variational principle

For the initial, mixed boundary value problem considered in Sect. 2, we now obtain a generalized Hamilton's principle in the form of the following theorem.

THEOREM. *If t_1 and t_2 are two arbitrary instants of time, then the field equations* (2.9)—(2.11) *form a set of necessary and sufficient conditions for the variational equation*

$$(5.1)\qquad \delta\int_{t_1}^{t_2}\Bigg[\int_m\Big\{T-\psi+\theta\Big(\frac{\partial\psi}{\partial\theta}+\frac{\alpha}{\theta_0}r\Big)+F_iu_i+M_i\phi_i\Big\}dm+\int_{\partial B_1}T_iu_i\,dA+\int_{\partial B_2}L_i\phi_i\,dA-\alpha\int_{\partial B_3}Q\theta\,dA\Bigg]dt=0$$

to hold for arbitrary variations δu_i, $\delta\phi_i$ and $\delta\theta$ in u_i, ϕ_i and θ respectively, which, in addition to being compatible with the kinematic constraints, satisfy the conditions

$$(5.2)\qquad \delta u_i(x,t)=\delta\phi_i(x,t)=\delta\theta(x,t)=0\quad\text{for}\quad x\in B\quad\text{and}\quad t=t_1,t_2,$$

$$(5.3)\qquad \delta u_i=0\quad\text{on }\partial B_1^c,\qquad \delta\phi_i=0\quad\text{on}\quad \partial B_2^c,\qquad \delta\theta=0\quad\text{on }\partial B_3^c$$

and the functions F_i, M_i, r, $\frac{\partial\psi}{\partial\theta}$, T_i, L_i and Q, and t being kept unchanged.

Proof. From Eqs. (3.6) and (5.2), we obtain

$$(5.4)\qquad \delta\int_{t_1}^{t_2}dt\int_m T\,dm=-\int_{t_1}^{t_2}dt\int_B\varrho(\ddot{u}_i\,\delta u_i+J_{ij}\ddot{\phi}_j\,\delta\phi_i)\,dB.$$

Equations (2.1), (2.5)—(2.7), (2.19)—(2.21), (3.3), (5.2) and (5.3) together with the divergence theorem yield

$$(5.5)\qquad \delta\int_{t_1}^{t_2}dt\int_m\Big(\psi-\frac{\partial\psi}{\partial\theta}\theta\Big)dm=\int_{t_1}^{t_2}dt\int_m\Big(\frac{\partial\psi}{\partial e_{ij}}\delta e_{ij}+\frac{\partial\psi}{\partial\phi_{j,i}}\delta\phi_{j,i}+\frac{\partial\psi}{\partial\dot{\theta}}\delta\dot{\theta}+\frac{\partial\psi}{\partial\theta_{,i}}\delta\theta_{,i}\Big)dm=\int_{t_1}^{t_2}dt\Big[\int_{\partial B_1}T_i\,\delta u_i\,dA+\int_{\partial B_2}L_i\,\delta\phi_i\,dA-\alpha\int_{\partial B_3}Q\,\delta\theta\,dA$$
$$-\int_B\{A_{ijkl}(u_{l,kj}+\varepsilon_{lkm}\phi_{m,j})+B_{ijkl}\phi_{l,kj}+a_{ij}(\theta+\alpha\dot{\theta})_{,j}\}\,\delta u_i\,dB$$
$$+\int_B\{B_{klji}(u_{l,kj}+\varepsilon_{lkm}\phi_{m,j})+C_{ijkl}\phi_{l,kj}+c_{ji}(\theta+\alpha\dot{\theta})_{,j}+\varepsilon_{ijk}A_{jkrs}(u_{s,r}+\varepsilon_{srm}\phi_m)$$
$$+\varepsilon_{ijk}B_{jkrs}\phi_{s,r}+\varepsilon_{ijk}a_{jk}(\theta+\alpha\dot{\theta})\}\,\delta\phi_i\,dB-\alpha\int_B\{d\dot{\theta}+h\ddot{\theta}-2b_i\dot{\theta}_{,i}$$
$$-a_{ij}(\dot{u}_{j,i}+\varepsilon_{jik}\dot{\phi}_k)-c_{ij}\dot{\phi}_{j,i}-k_{ij}\theta_{,ji}\}\,\delta\theta\,dB\Big].$$

With the aid of Eqs. (5.4) and (5.5), the variational equation (5.1) reduces to

$$(5.6)\qquad \int_{t_1}^{t_2}dt\int_B\Big[\{A_{ijkl}(u_{l,kj}+\varepsilon_{lkm}\phi_{m,j})+B_{ijkl}\phi_{l,kj}+a_{ij}(\theta+\alpha\dot{\theta})_{,j}+\varrho(F_i-\ddot{u}_i)\}\,\delta u_i$$
$$+\{B_{klji}(u_{l,kj}+\varepsilon_{lkm}\phi_{m,j})+C_{ijkl}\phi_{l,kj}+C_{ji}(\theta+\alpha\dot{\theta})_{,j}+\varepsilon_{ijk}A_{jkrs}(u_{s,r}+\varepsilon_{srm}\phi_m)$$
$$+\varepsilon_{ijk}B_{jkrs}\phi_{s,r}+\varepsilon_{ijk}a_{jk}(\theta+\alpha\dot{\theta})+\varrho(M_i-J_{ij}\ddot{\phi}_j)\}\,\delta\phi_i$$
$$+\alpha\Big\{\frac{\varrho r}{\theta_0}-d\dot{\theta}-h\ddot{\theta}+2b_i\dot{\theta}_{,i}+a_{ij}(\dot{u}_{j,i}+\varepsilon_{jik}\dot{\phi}_k)+c_{ij}\dot{\phi}_{j,i}+k_{ij}\theta_{,ji}\Big\}\delta\theta\Big]dB=0.$$

Obviously, this equation holds if and only if Eqs. (2.9)–(2.11) are satisfied.

This completes the proof.

It may be noted that whereas the variational principles of the Hamilton-type obtained in the conventional thermoelasticity theories involve two functionals (see, for example, [25]), the principle obtained above involves just one. In the absence of polar effects this variational principle reduces to the one obtained in [24].

6. Reciprocal theorem

We consider two initial, mixed boundary value problems associated with the two systems:

$$\{(F_i^{(\eta)}, M_i^{(\eta)}, r^{(\eta)}); (T_i^{(\eta)}, U_i^{(\eta)}, L_i^{(\eta)}, \Phi_i^{(\eta)}, \Theta^{(\eta)}, Q^{(\eta)}); (u_i^{(\eta)}, \phi_i^{(\eta)}, \theta^{(\eta)})\}, \qquad \eta = 1, 2$$

and suppose that $t_{ij}^{(\eta)}$, $m_{ij}^{(\eta)}$, $q_i^{(\eta)}$ are the corresponding stresses, couple stresses and heat flux. For η, $\mu = 1, 2$, if

$$\begin{aligned}(6.1)\quad \Sigma_{\eta\mu} = \int\limits_B \Big[&\varrho(F_i^{(\eta)} * \tilde{u}_i^{(\mu)} + M_i^{(\eta)} * \tilde{\phi}_i^{(\mu)}) - \frac{\varrho}{\theta_0}\{(r^{(\eta)} * \theta^{(\mu)}) + \alpha(r^{(\eta)} * \tilde{\theta}^{(\mu)})\} \\ &- 2b_i\{(\theta_{,i}^{(\eta)} * \tilde{\theta}^{(\mu)}) + \alpha(\theta_{,i}^{(\eta)} * \tilde{\tilde{\theta}}^{(\mu)})\}\Big] dB + \int\limits_{\partial B_1} (T_i^{(\eta)} * \tilde{u}_i^{(\mu)}) dA \\ &- \int\limits_{\partial B_1^c} (U_i^{(\eta)} * \tilde{t}_{ij}^{(\mu)} n_j) dA + \int\limits_{\partial B_2} (L_i^{(\eta)} * \tilde{\phi}_i^{(\mu)}) dA - \int\limits_{\partial B_2^c} (\Phi_i^{(\eta)} * \tilde{m}_{ji}^{(\mu)} n_j) dA \\ &+ \int\limits_{\partial B_3} \{(Q^{(\eta)} * \theta^{(\mu)}) + \alpha(Q^{(\eta)} * \tilde{\theta}^{(\mu)})\} dA - \frac{1}{\theta_0} \int\limits_{\partial B_3^c} \{(\Theta^{(\eta)} * q_i^{(\mu)}) + \alpha(\Theta^{(\eta)} * \tilde{q}_i^{(\mu)})\} n_i dA,\end{aligned}$$

where

$$f * g = \int\limits_0^t f(x, t-t_0) g(x, t_0) dt_0,$$

$$(6.2)\qquad f * \tilde{g} = \int\limits_0^t f(x, t-t_0) \frac{\partial g}{\partial t_0}(x, t_0) dt_0,$$

$$f * \tilde{\tilde{g}} = \int\limits_0^t f(x, t-t_0) \frac{\partial^2 g}{\partial t_0^2}(x, t_0) dt_0,$$

we prove the reciprocal theorem that

$$(6.3)\qquad \Sigma_{12} = \Sigma_{21}$$

P r o o f. By hypothesis, the functions associated with the two problems considered are governed by the field equations (2.9)—(2.11) and the boundary conditions (2.19)—(2.21). Taking the Laplace transform of these equations, under the initial conditions (2.18), we obtain the following equations for the transformed functions, by using (2.1).

$$(6.4)\qquad A_{ijkl}\bar{e}_{kl,j}^{(\eta)} + B_{ijkl}\bar{\phi}_{l,kj}^{(\eta)} + (1+\alpha\omega) a_{ij}\bar{\theta}_{,j}^{(\eta)} + \varrho\bar{F}_i^{(\eta)} = \varrho\omega^2\bar{u}_i^{(\eta)},$$

$$\text{(6.5)}\quad B_{klji}\bar{e}^{(\eta)}_{kl,j}+C_{jikl}\bar{\phi}^{(\eta)}_{l,kj}+(1+\alpha\omega)c_{ji}\bar{\theta}^{(\eta)}_{,j}+\varepsilon_{ijk}\{A_{jkrs}\bar{e}^{(\eta)}_{rs}+B_{jkrs}\bar{\phi}^{(\eta)}_{s,r}+(1+\alpha\omega)a_{jk}\bar{\theta}^{(\eta)}\}+\varrho\bar{M}^{(\eta)}_i=\varrho\omega^2 J_{ij}\bar{\phi}^{(\eta)}_j,$$

$$\text{(6.6)}\quad \frac{\varrho}{\theta_0}\bar{r}^{(\eta)}+k_{ij}\bar{\theta}^{(\eta)}_{,ji}-\omega\{(d+h\omega)\bar{\theta}^{(\eta)}+2b_i\bar{\theta}^{(\eta)}_{,i}+a_{ij}\bar{e}^{(\eta)}_{ij}+c_{ij}\bar{\phi}^{(\eta)}_{j,i}\}=0,$$

$$\text{(6.7)}\quad \bar{t}^{(\eta)}_{ij}n_j=\bar{T}^{(\eta)}_i \quad \text{on } \partial B_1,\qquad \bar{u}^{(\eta)}_i=\bar{U}^{(\eta)}_i \quad \text{on } \partial B_1^c,$$

$$\text{(6.8)}\quad \bar{m}^{(\eta)}_{ji}n_j=\bar{L}^{(\eta)}_i \quad \text{on } \partial B_2,\qquad \bar{\phi}^{(\eta)}_i=\bar{\Phi}^{(\eta)} \quad \text{on } \partial B_2^c,$$

$$\text{(6.9)}\quad \bar{q}^{(\eta)}_i n_i=\theta_0\bar{Q}^{(\eta)} \quad \text{on } \partial B_3,\qquad \bar{\theta}^{(\eta)}=\bar{\Theta}^{(\eta)} \quad \text{on } \partial B_3^c.$$

Also Eqs. (2.5)—(2.7) yield

$$\text{(6.10)}\quad \bar{t}^{(\eta)}_{ij}=A_{ijkl}\bar{e}^{(\eta)}_{kl}+B_{ijkl}\bar{\phi}^{(\eta)}_{l,k}+(1+\alpha\omega)a_{ij}\bar{\theta}^{(\eta)},$$

$$\text{(6.11)}\quad \bar{m}^{(\eta)}_{ij}=B_{klij}\bar{e}^{(\eta)}_{kl}+C_{ijkl}\bar{\phi}^{(\eta)}_{l,k}+(1+\alpha\omega)c_{ij}\bar{\theta}^{(\eta)},$$

$$\text{(6.12)}\quad \bar{q}^{(\eta)}_i=-\theta_0(\omega b_i\bar{\theta}^{(\eta)}+k_{ij}\bar{\theta}^{(\eta)}_{,j}).$$

In all these equations, $\bar{f}=\bar{f}(x,\omega)$ denotes the Laplace transform of $f(x,t)$.
From Eqs. (6.4)—(6.6) and (6.10)—(6.12), we obtain

$$\text{(6.13)}\quad \{\bar{t}^{(2)}_{ij}\bar{u}^{(1)}_i+\bar{m}^{(2)}_{ji}\bar{\phi}^{(1)}_i-\bar{t}^{(1)}_{ij}\bar{u}^{(2)}_i-\bar{m}^{(1)}_{ji}\bar{\phi}^{(2)}_i\}_{,j}=\varrho(\bar{F}^{(1)}_i\bar{u}^{(2)}_i+\bar{M}^{(1)}_i\bar{\phi}^{(2)}_i-\bar{F}^{(2)}_i\bar{u}^{(1)}_i-\bar{M}^{(2)}_i\bar{\phi}^{(1)}_i)-(1+\alpha\omega)\{a_{ij}(\bar{\theta}^{(1)}\bar{e}^{(2)}_{ji}-\bar{\theta}^{(2)}\bar{e}^{(1)}_{ji})+c_{ij}(\bar{\theta}^{(1)}\bar{\phi}^{(2)}_{j,i}-\bar{\theta}^{(2)}\bar{\phi}^{(1)}_{j,i})\},$$

$$\text{(6.14)}\quad (\bar{q}^{(1)}_i\bar{\theta}^{(2)}-\bar{q}^{(2)}_i\bar{\theta}^{(1)})_{,i}=\varrho(\bar{r}^{(1)}\bar{\theta}^{(2)}-\bar{r}^{(2)}\bar{\theta}^{(1)})+\theta_0\omega\{a_{ij}(\bar{e}^{(1)}_{ji}\bar{\theta}^{(2)}-\bar{e}^{(2)}_{ji}\bar{\theta}^{(1)})+c_{ij}(\bar{\phi}^{(1)}_{j,i}\bar{\theta}^{(2)}-\bar{\phi}^{(2)}_{j,i}\bar{\theta}^{(1)})+2b_i(\bar{\theta}^{(1)}_{,i}\bar{\theta}^{(2)}-\bar{\theta}^{(2)}_{,i}\bar{\theta}^{(1)})\}.$$

With the aid of the divergence theorem and the boundary conditions (6.7)—(6.9). Eqs. (6.13) and (6.14) yield

$$\text{(6.15)}\quad \int_B\Big[\varrho\omega(\bar{F}^{(1)}_i\bar{u}^{(2)}_i+\bar{M}^{(1)}_i\bar{\phi}^{(2)}_i-\bar{F}^{(2)}_i\bar{u}^{(1)}_i-\bar{M}^{(2)}_i\bar{\phi}^{(1)}_i)-\frac{\varrho}{\theta_0}(1+\alpha\omega)(\bar{r}^{(1)}\bar{\theta}^{(2)}-\bar{r}^{(2)}\bar{\theta}^{(1)})-2\omega b_i(1+\alpha\omega)(\bar{\theta}^{(1)}_{,i}\bar{\theta}^{(2)}-\bar{\theta}^{(2)}_{,i}\bar{\theta}^{(1)})\Big]dB$$
$$+\omega\Big[\int_{\partial B_1}(\bar{T}^{(1)}_i\bar{u}^{(2)}_i-\bar{T}^{(2)}_i\bar{u}^{(1)}_i)dA+\int_{\partial B_1^c}(\bar{t}^{(1)}_{ij}\bar{U}^{(2)}_i-\bar{t}^{(2)}_{ij}\bar{u}^{(1)}_i)n_j\,dA$$
$$+\int_{\partial B_2}(\bar{L}^{(1)}_i\bar{\phi}^{(2)}_i-\bar{L}^{(2)}_i\bar{\phi}^{(1)}_i)dA+\int_{\partial B_2^c}(\bar{m}^{(1)}_{ji}\bar{\Phi}^{(2)}_i-\bar{m}^{(2)}_{ji}\bar{\Phi}^{(1)}_i)n_j\,dA$$
$$+\frac{1+\alpha\omega}{\theta_0}\Big[\theta_0\int_{\partial B_3}(\bar{Q}^{(1)}\bar{\theta}^{(2)}-\bar{Q}^{(2)}\bar{\theta}^{(1)})dA+\int_{\partial B_3^c}(\bar{q}^{(1)}_i\bar{\Theta}^{(2)}-\bar{q}^{(2)}_i\bar{\Theta}^{(1)})n_i\,dA\Big]=0.$$

Inverting this equation by using the convolution theorem for Laplace transforms, we obtain the desired equation (6.3).

If we neglect the micropolar effects, this theorem reduces to that obtained in [24]. If we also set $\alpha = b_i = 0$ in this theorem we recover the reciprocal theorem of [23], obtained for isotropic materials.

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