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A useful analogy between equations of linear elasticity and rigid-plasticity of void-containing metals

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A FORMAL analogy between equations describing two different classes of material behaviour is demonstrated. The numerical usefulness of the analogy in the analysis of plastic flow problems is indicated.

IT IS KNOWN that there exists a formal analogy between equations of linear incompressible elasticity and equations describing plastic flow. The equations which express this analogy can be written in a compact form as

Plastic flow — Incompressible elasticity

$$\sigma_{ij,j}+b_i = 0,$$

(1)
$$\dot{\varepsilon}_{ij} = \frac{1}{2\mu}(\sigma_{ij}-\delta_{ij}p) \leftrightarrow \varepsilon_{ij} = \frac{1}{2G}(\sigma_{ij}-\delta_{ij}p)$$
$$\dot{\varepsilon}_{ij} = \frac{1}{2}(\dot{u}_{i,j}+\dot{u}_{j,i}) \leftrightarrow \varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i})$$
$$\dot{u}_i \leftrightarrow u_i$$

in which the standard notation is employed with p being the mean normal stress and $\mu = \sigma_y/3\dot{\varepsilon}$, σ_y is the uniaxial stress of the material and $\dot{\varepsilon}$ is the strain rate invariant $\dot{\varepsilon} = \sqrt{(2/3)\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}}$.

The analogy has a great computational potential since it allows to treat plastic flow problems using computer software developed for linear elasticity, [1]. To do so one has to simply allow the elastic shear modulus G to be a function of the stress or strain level and to interpret the displacements u_i as the instantenous velocities $\dot{u}_i$. In this note we show that a recently developed, [2, 3, 4] and widely used, [5, 6] theory describing plastic flow of void-containing metals can be reformulated to display the same kind of analogy with respect to compressible elasticity. The description we refer ourselves to is known in the literature as the Gurson's model. It has been worked out starting with a simple rigid-plastic void growth analysis and has now the formal structure of nonassociated plasticity equations with plastic dilatancy effect taken appropriately into account. The ductile fracture process is described as an apparent loss of active material volume with a corresponding decay of the average macroscopic stresses. The yield condition for a randomly voided material with spherical (for $3D$ problems) or circular cylindrical (for plane stress or plane strain problems) voids is assumed as

(2)
$$\Phi(\sigma_{ij}, \sigma_M, f) = \frac{3}{2}\frac{\sigma^D_{ij}\sigma^D_{ij}}{\sigma^2_M} + 2f\cosh\left(\frac{\sigma_{kk}}{2\sigma_M}\right) - (1+f^2) = 0,$$

where σ_{ij} — macroscopic Cauchy stress, $\sigma_{kk} = 3p$, σ_M — microscopic tensile yield limit of the matrix material, f — void volume fraction.

The summation convention is employed. The matrix material is assumed incompressible. For the porosity parameter $f = 0$ the condition (2) reduces to the classical Huber–Mises yield condition of the form

$$\Phi(\sigma_{ij}, \sigma_Y) = \frac{3}{2}\frac{\sigma^D_{ij}\sigma^D_{ij}}{\sigma^2_Y} - 1 = 0 \tag{3}$$

with $\sigma_Y = \sigma_M$ being the current tensile yield limit. According to concepts discussed in [2−6], the change of the void volume fraction during the increment of deformation may be taken as

$$\dot{f} = \dot{f}_{\text{growth}} + \dot{f}_{\text{nucleation}} \tag{4}$$

where it may be, for instance, approximately assumed that

$$\dot{f}_{\text{growth}} = (1-f)\,\dot{\varepsilon}_{kk}, \tag{5}$$

$$\dot{f}_{\text{nucloation}} = \frac{\hat{K}}{\sigma_M}\left(\dot{\sigma}_M + \frac{\dot{\sigma}_{kk}}{3}\right), \tag{6}$$

the material parameter $\hat{K}$ being the void volume fraction of particles converted to voids per unit fractional increase in stress. More complex and accurate expressions can obviously be used instead of Eqs. (5) and (6).

After some routine calculations it is possible to arrive at the typical nonassociated flow rule, [2, 5]

$$\dot{\varepsilon}_{ij} = \frac{1}{\bar{\zeta}}\, {}_{(1)}s_{ij}\left({}_{(2)}s_{kl}\,\dot{\sigma}_{kl}\right) \tag{7}$$

in which

$$\begin{gathered} {}_{(1)}s_{ij} = \frac{3}{2}\frac{\sigma^D_{ij}}{\sigma_M} + \beta\delta_{ij}, \qquad {}_{(2)}s_{ij} = \frac{3}{2}\frac{\sigma^D_{ij}}{\sigma_M} + \mu\delta_{ij}, \\ \beta = \frac{fs}{2}\cdot \mu = \beta + \frac{\hat{K}}{6}\frac{\partial\Phi}{\partial f} = \beta + \hat{K}\frac{c-f}{3}. \\ \bar{\zeta} = \zeta\,\frac{(\omega + f\Sigma s)^2}{1-f} - \frac{(c-f)\sigma_M}{2}\left[3f(1-f)s + 2\frac{\hat{K}}{\sigma_M}\,\zeta\,\frac{\omega + f\Sigma s}{1-f}\right], \end{gathered} \tag{8}$$

σ^D_{ij} — Cauchy stress deviator; $\bar{\zeta}$ — effective hardening (or softening) modulus for voided material; ζ — classical isotropic hardening parameter for the matrix material, $\zeta = \dfrac{EE_T}{E-E_T}$; E — Young modulus; E_T — slope of the stress-strain curve,

$$\omega = \frac{\bar{\sigma}^2}{\sigma^2_M} = \frac{\frac{3}{2}\sigma^D_{ij}\sigma^D_{ij}}{\sigma^2_M} = 1 - 2fc + f^2 \tag{9}$$

is the square of the ratio of the macroscopic to microscopic equivalent yield stress, and we have used the shortened notation

$$\Sigma = \frac{\sigma_{kk}}{2\sigma_M}, \quad s = \sinh\Sigma, \quad c = \cosh\Sigma. \tag{10}$$

It is easily seen that for the classical plastic material we have to put $f = 0$, $\hat{K} = 0$ and $\omega = 1$ and then the classical associated flow rule is recovered from Eq. (7) in the form

$$\dot{\varepsilon}_{ij} = \frac{1}{\zeta} s_{ij}(s_{kl}\dot{\sigma}_{kl}), \quad s_{ij} = \frac{3}{2}\frac{\sigma^D_{ij}}{\sigma}. \tag{11}$$

After further elementary algebra the flow rule (11) may be reduced to another known form

$$\dot{\varepsilon}_{ij} = \frac{1}{2\mu}\sigma^D_{ij}, \quad \mu = \bar{\sigma}/3\dot{\bar{\varepsilon}}, \quad \zeta = \frac{\dot{\bar{\sigma}}}{\dot{\bar{\varepsilon}}}. \tag{12}$$

Similar calculations carried out for the porous material lead instead of Eq. (12) to the flow rule of the form

$$\dot{\varepsilon}_{ij} = \frac{1}{\bar{\zeta}}\frac{3}{2\sigma_M}(\omega\dot{\sigma}_M + \mu\dot{\sigma}_{kk} + 0.5\dot{\omega}\sigma_M)\left[\sigma_{ij} + \left(\frac{f}{2}\frac{\sinh\Sigma}{\Sigma} - 1\right)p\delta_{ij}\right] \tag{13}$$

or

$$\dot{\varepsilon}_{ij} = \frac{1}{2G^*}[\sigma_{ij} - 2\nu^* p\delta_{ij}], \tag{14}$$

where

$$G^* = \frac{\sigma_M\bar{\zeta}}{3(\omega\dot{\sigma}_M + \mu\dot{\sigma}_{kk} + 0.5\dot{\omega}\sigma_M)}, \quad \nu^* = \frac{1}{2}\left(1 - \frac{f}{2}\frac{\sinh\Sigma}{\Sigma}\right). \tag{15}$$

The transition to classical plasticity is again seen clearly by comparing Eqs. (14) and (15) for $f = 0$, $\omega = 1$, $\mu = 0$ and Eq. (12).

Equation (14) allows to present the discussed analogy in the following form:

Plastic flow in void-containing metals	Linear elasticity

$$\sigma_{ij,j} + b_i = 0$$

$$\dot{\varepsilon}_{ij} = \frac{1}{2G^*}(\sigma_{ij} - 2\nu^* p\delta_{ij}) \leftrightarrow \varepsilon_{ij} = \frac{1}{2G}(\sigma_{ij} - 2\nu p\delta_{ij})$$

$$\dot{\varepsilon}_{ij} = \frac{1}{2}(\dot{u}_{ij} + \dot{u}_{ji}) \leftrightarrow \varepsilon_{ij} = \frac{1}{2}(u_{ij} + u_{ji})$$

$$\dot{u}_i \leftrightarrow u_i.$$

It must be emphasized that the „material coefficients" G^* and ν^* are in fact functions of the current state variables with G^* depending also on their rates. Thus, an iterative procedure is clearly needed to full expoit existing finite element codes for linear elastic problems.

However, the computational usefulness of this analogy can hardly be overestimated. The reader is referred to [7] for details of the numerical approach and illustrative examples from the field of axisymmetric metal sheet forming.

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