

On the phenomenological theory of ferromagnetism

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FOR A RIGID ferromagnetic medium field equations and jump conditions are derived from the global balance laws of micromagnetic theory. Introduction of a dissipation potential allows the unified description of dissipative effects such as damping processes and conduction phenomena. The results are specialized for a uniaxial ferromagnet.

Dla sztywnego ośrodka ferromagnetycznego wyprowadzono równania pola oraz warunki dla skoków z ogólnych równań zachowania teorii mikromagnetyzmu. Wprowadzenie potencjału dysypatywnego umożliwiło jednolity opis efektów rozpraszania energii takich, jak procesy tłumienia i zjawiska przewodzenia. Otrzymano wyniki dla przypadku jednoosiowego ferromagnetu.

Для жесткой ферромагнитной среды выведены уравнения поля, а также условия для скачков из общих уравнений сохранения теории микромагнетизма. Введение диссипативного потенциала дало возможность однородного описания диссипативных эффектов, таких как процессы затухания и явления проводимости. Общие результаты специфицированы для случая одноосного ферромагнетика.

1. Introduction

AMONG magnetic substances ferromagnetic materials exhibit strong magnetic polarisation, even at small or vanishing applied magnetic fields, assuming that the temperature is sufficiently low. This behaviour is based on a strong interaction between magnetic moments on an atomic scale which are, essentially, of quantum mechanical origin and are called "exchange interactions". A general description of the behaviour of ferromagnetic bodies should, therefore, make use of a microscopic model.

Here, we shall confine our attention to a phenomenological treatment which is often sufficient to describe macroscopic effects. Starting with global balance laws, local relations and jump conditions are derived for a rigid, ferromagnetic medium at rest in a Lorentz-frame. The introduction of a dissipation potential allows a unified description of damping processes within the spin continuum and dissipative effects based on the transport of electric charge as well as the derivation of symmetry and reciprocity relations. The constitutive equations are specialized for a uniaxial ferromagnet; a small-perturbation susceptibility tensor is obtained.

2. Global and local balance equations. Jump conditions. Balance of energy

We consider a rigid, ferromagnetic body at rest with respect to a Lorentz-frame of reference, and delimitate a macroscopic part with volume v by a closed surface ∂v . If ϵ

denotes the internal energy density of the medium, the energy balance can be written in the form [1]

$$(2.1) \quad \frac{d}{dt} \int_v \varepsilon dv = \oint_{\partial v} (\dot{\mathbf{J}} \cdot \mathbf{t} - \mathbf{q}) \cdot \mathbf{n} da + \int_v (r + \dot{\mathbf{J}} \cdot \mathbf{H} + \mathbf{E} \cdot \mathbf{S}) dv.$$

The time rate of change of internal energy enclosed in v is caused by supply through the boundary and within the volume. \mathbf{n} is the external surface normal and \mathbf{q} the density of heat flux. The magnetic polarisation field \mathbf{J} is, microscopically, constituted by magnetic moments coupled together by exchange interactions which vanish rapidly with increasing distance (spin-spin coupling). Within the frame of a continuum theory, such interactions can be described as being of the contact type and are, according to Cauchy's principle, representable by a second-rank tensor, the exchange tensor \mathbf{t} . Hence, we can interpret the vector $\mathbf{t} \cdot \mathbf{n}$ as the magnetic contact field strength; the corresponding power per unit area is $\dot{\mathbf{J}} \cdot \mathbf{t} \cdot \mathbf{n}$. (A superposed dot denotes partial derivation with respect to time). r is the external volume supply of heat. With the electromagnetic field, energy is exchanged in the form of polarisation power $\dot{\mathbf{J}} \cdot \mathbf{H}$ (\mathbf{H} is the magnetic field strength) and power of electric conduction current density \mathbf{S} in the electric field \mathbf{E} . Electric polarisation and charges are neglected.

Thermodynamically, we consider the system "ferromagnetic body" as being distinct from the system "electromagnetic field"; the links between these two systems are $\dot{\mathbf{J}} \cdot \mathbf{H}$ and $\mathbf{S} \cdot \mathbf{E}$. The energy balance of the system "electromagnetic field" is Poynting's theorem in the form

$$\frac{d}{dt} \int_v \left(\frac{1}{2} \mu_0 H^2 + \frac{1}{2} \varepsilon_0 E^2 \right) dv = \oint_{\partial v} -(\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} da + \int_v (-\dot{\mathbf{J}} \cdot \mathbf{H} - \mathbf{S} \cdot \mathbf{E}) dv,$$

where $(\mu_0 H^2 + \varepsilon_0 E^2)/2$ has to be interpreted as electromagnetic energy density, $\mathbf{E} \times \mathbf{H}$ as electromagnetic energy flux density, and $-\dot{\mathbf{J}} \cdot \mathbf{H} - \mathbf{S} \cdot \mathbf{E}$ as the volume supply of electromagnetic energy.

Since the global energy balance is supposed to be valid for arbitrary regions of space, we obtain the local form (in Cartesian tensor notation)

$$(2.2) \quad \dot{\varepsilon} = r - q_{i,i} + S_i E_i + t_{ij} \dot{J}_{i,j} + (H_i + t_{ij,j}) \dot{J}_i,$$

and on singular surfaces

$$[[\dot{J}_i t_{ij} - q_j]] n_j = 0,$$

if the integrands in Eq. (2.1) are bounded. The symbol $[[A]] = A^+ - A^-$ denotes the jump of a quantity A on a surface with normal directed from $-$ to $+$.

Spin balance

The magnetic moment of the body is connected with an internal angular momentum via the gyromagnetic ratio γ . We suppose the magnetic polarisation to be distributed in a spin continuum, linked with the electromagnetic field by a couple density $\mathbf{J} \times_L \mathbf{H}$, and with the lattice continuum by a couple density $\mathbf{J} \times_L \mathbf{H}$ and a damping moment $-\mathbf{R}$. Parts

of the spin continuum act on each other through exchange forces of the contact type and cause a surface density of couples $\mathbf{J} \times \mathbf{t} \cdot \mathbf{n}$. If changes of external angular momentum are compensated by mechanical torques, we have

$$(2.3) \quad \frac{d}{dt} \int_v \frac{1}{\gamma} \mathbf{J} dv = \oint_{\partial v} \mathbf{J} \times \mathbf{t} \cdot \mathbf{n} da + \int_v [\mathbf{J} \times (\mathbf{H} + {}_L\mathbf{H}) - \mathbf{R}] dv.$$

The local magnetic field strength ${}_L\mathbf{H}$ describes anisotropy effects as spin-orbit coupling and interactions with magnetic moments of higher order. The corresponding local form is

$$(2.4) \quad \frac{1}{\gamma} \dot{J}_i = \varepsilon_{ijk} J_j (H_k + {}_L H_k + t_{kl,i}) + \varepsilon_{ijk} J_{j,l} t_{kl} - R_i,$$

and the jump condition on singular surfaces

$$\varepsilon_{ijk} \llbracket J_j t_{kl} \rrbracket n_l = 0.$$

Clausius-Duhem inequality

The Clausius-Duhem inequality

$$(2.5) \quad \frac{d}{dt} \int_v S dv \geq \oint_{\partial v} -\frac{1}{T} \mathbf{q} \cdot \mathbf{n} da + \int_v \frac{1}{T} r dv$$

represents a form of the second law of thermodynamics ([2, 3]) stating that the temporal increase of entropy enclosed in v with the density S is never smaller than the sum of entropy supply through surface and volume. T denotes the thermodynamic temperature assumed to be positive, \mathbf{q}/T the density of entropy flux and r/T the external volume supply of entropy. Passing over to the local relation and jump condition

$$(2.6) \quad \dot{S} \geq \frac{r}{T} - \left(\frac{1}{T} q_i \right)_{,i} \quad \text{and} \quad \llbracket \frac{1}{T} q_i \rrbracket n_i \geq 0$$

we have again supposed boundedness of integrands.

The system of equations given so far is extremely underdetermined. To obtain a complete system, besides Maxwell's equations of the electromagnetic field, a set of constitutive equations has to be added.

We define the free energy density by $F = \varepsilon - TS$, supposing this function to depend only on the magnetic polarisation, its gradient, and the thermodynamic temperature,

$$(2.7) \quad F = F(J_i; J_{i,j}, T),$$

and introduce it into the local energy balance (2.2):

$$(2.8) \quad \left(\frac{\partial F}{\partial J_i} - H_i - t_{ij,j} \right) \dot{J}_i + \left(\frac{\partial F}{\partial J_{i,j}} - t_{ij} \right) \dot{J}_{i,j} + \left(\frac{\partial F}{\partial T} + S \right) \dot{T} = r - q_{i,i} + S_i E_i - T \dot{S}.$$

In accordance with the usual assumptions of the micromagnetic theory (e.g. [4], [5]), we set the magnitude of magnetic polarisation equal to the saturation polarisation and assume both temporal and spatial temperature variations to be sufficiently small, so that

$$(2.9) \quad J_i J_i = J_s^2 = \text{const.}, \quad J_i \dot{J}_i = 0, \quad J_i J_{i,j} = 0.$$

The variables $J_{i,j}$ serve to describe the exchange interactions [4]. Since the damping moment \mathbf{R} is, in essence, caused by conduction electrons, we take it as independent of $\mathbf{J}_{i,j}$ and assume, after transvecting the spin equation (2.4) by J_i ,

$$(2.10) \quad \varepsilon_{ijk} J_i J_{j,l} t_{kl} = 0 \quad \text{and} \quad J_i R_l = 0.$$

Next, the field strength \mathbf{H} is eliminated from Eq. (2.8) by means of the spin equation. For this purpose, we split \mathbf{H} into components perpendicular and parallel to \mathbf{J} , $\mathbf{H} = \mathbf{H}_\perp + \mathbf{H}_\parallel$, take the cross product of Eq. (2.4) and \mathbf{J} , and obtain with Eq. (2.9)

$$H_{\perp i} = -\frac{1}{J_s^2} \left[\frac{1}{\gamma} \varepsilon_{ijk} J_j (\dot{J}_k + R_k) - J_{i,l} J_k t_{kl} - J_i J_j (\mathcal{L} H_j + t_{jl,i}) \right] - (\mathcal{L} H_i + t_{ij,j}).$$

Transvection of this equation with $\dot{\mathbf{J}}$ renders, since $\dot{\mathbf{J}} \cdot \mathbf{H}_\perp = \dot{\mathbf{J}} \cdot \mathbf{H}$,

$$(2.11) \quad (H_i + t_{ij,j}) \dot{J}_i = \left[\frac{1}{J_s^2} (J_j t_{jk} J_{i,k} - \varepsilon_{ijk} J_j R_k) - \mathcal{L} H_i \right] \dot{J}_i.$$

Neither the spin equation (2.3) nor the energy equation (2.1) contains components of the contact field strength $\mathbf{t} \cdot \mathbf{n}$ parallel to the magnetic polarisation. Without physical restriction we can, therefore, demand

$$(2.12) \quad J_i t_{ij} = 0,$$

and conclude from this and Eqs. (2.8) and (2.11)

$$(2.13) \quad \left(\frac{\partial F}{\partial J_i} + \mathcal{L} H_i + \frac{1}{J_s^2} \varepsilon_{ijk} J_j R_k \right) \dot{J}_i + \left(\frac{\partial F}{\partial J_{i,j}} - t_{ij} \right) \dot{J}_{i,j} + \left(\frac{\partial F}{\partial T} + S \right) \dot{T} = r - q_{i,i} + S_i E_i - T \dot{S}.$$

We define the density of entropy production by

$$(2.14) \quad \psi = \dot{S} - \frac{1}{T} r + \frac{1}{T} q_{i,i} - \frac{1}{T^2} q_i T_{,i}$$

and write the Clausius-Duhem inequality (2.6) in the form

$$(2.15) \quad \psi \geq 0$$

or, after elimination of the external heat supply r ,

$$(2.16) \quad \psi T = S_i E_i - \frac{1}{T} q_i T_{,i} - \left(\frac{\partial F}{\partial T} + S \right) \dot{T} - \left(\frac{\partial F}{\partial J_{i,j}} - t_{ij} \right) \dot{J}_{i,j} - \left(\frac{\partial F}{\partial J_i} + \mathcal{L} H_i + \frac{1}{J_s^2} \varepsilon_{ijk} J_j R_k \right) \dot{J}_i \geq 0.$$

3. General constitutive equations. Dissipation potential

The variables of the free energy density function have been adjusted by the relation (2.7). With respect to the other constitutive functions we presume the following list which will be discussed later:

$$\begin{aligned} \mathcal{L} H_i &= \mathcal{L} H_i(J_i, J_{i,j}, T) && \text{local anisotropy field strength,} \\ t_{ij} &= t_{ij}(J_i, J_{i,j}, T) && \text{exchange tensor,} \end{aligned}$$

$$(3.1) \quad \begin{aligned} S &= S(J_i, J_{i,j}, T) && \text{entropy density,} \\ q_i &= q_i(J_i, E_i, T, T_{,i}) && \text{heat flux density,} \\ S_i &= S_i(J_i, E_i, T, T_{,i}) && \text{electric current density,} \\ R_i &= R_i(J_i, \dot{J}_i, T) && \text{damping moment.} \end{aligned}$$

The inequality (2.16) is, therefore, linear in \dot{T} and $\dot{J}_{i,j}$ and, since Eq. (2.16) must hold for arbitrary processes, the coefficients of these quantities vanish. We obtain

$$(3.2) \quad S = -\frac{\partial F}{\partial T}, \quad t_{ij} = \frac{\partial F}{\partial J_{i,j}},$$

and the remaining inequality

$$(3.3) \quad \psi T = E_i S_i - T_{,i} \frac{1}{T} q_i - \dot{J}_i \left(\frac{\partial F}{\partial J_i} + {}_L H_i + \frac{1}{J_s^2} \varepsilon_{ijk} J_j R_k \right) \geq 0.$$

Now, the arguments of the free energy density function are summarized in a vector

$$(3.4) \quad \mathbf{Y} = \{J_i, J_{i,j}, T\};$$

furthermore, two other vectors \mathbf{X} and \mathbf{I} are introduced by

$$(3.5) \quad \mathbf{X} = \{E_i, T_{,i}, \dot{J}_i\}$$

and

$$(3.6) \quad \mathbf{I} = \left\{ S_i, -\frac{1}{T} q_i, -\left(\frac{\partial F}{\partial J_i} + {}_L H_i + \frac{1}{J_s^2} \varepsilon_{ijk} J_j R_k \right) \right\},$$

$$\mathbf{I} = \mathbf{I}(\mathbf{Y}; \mathbf{X}).$$

The inequality (3.3) can be written in the form

$$(3.7) \quad \psi T = \mathbf{X} \cdot \mathbf{I} \geq 0 \quad \text{or} \quad \psi(\mathbf{Y}; \mathbf{X}) \geq 0 \quad \text{with} \quad \psi(\mathbf{Y}; \mathbf{0}) = 0;$$

the entropy production has, therefore, a minimum at $\mathbf{X} = \mathbf{0}$. If we assume the vector function $\mathbf{I}(\mathbf{Y}; \mathbf{X})$ as being continuously differentiable with respect to arguments \mathbf{X} for all \mathbf{Y} , this minimum is analytic, hence,

$$(3.8) \quad \mathbf{I}(\mathbf{Y}; \mathbf{0}) = \mathbf{0} \quad \text{for all} \quad \mathbf{Y},$$

especially,

$$(3.9) \quad \begin{aligned} S_i(J_i, E_i = 0, T, T_{,i} = 0) &= 0 \\ q_i(J_i, E = 0, T, T_{,i} = 0) &= 0 \end{aligned} \quad \text{for all } J_i, T.$$

Electric and thermal flux densities are zero if the electric field strength and temperature gradient vanish together, independent of special values of magnetic polarisation and temperature.

A vector function $\mathbf{I}(\mathbf{Y}; \mathbf{X})$ on abstract vector spaces is uniquely representable in the form [6]

$$(3.10) \quad \mathbf{I}(\mathbf{Y}; \mathbf{X}) = \nabla_{\mathbf{x}} \phi(\mathbf{Y}; \mathbf{X}) + \mathbf{U}(\mathbf{Y}; \mathbf{X}),$$

where the symbol ∇_x denotes partial differentiation of the twice continuously differentiable scalar-valued function ϕ with respect to components of the vector \mathbf{X} . The vector-valued function \mathbf{U} fulfills the requirements

$$(3.11) \quad \mathbf{X} \cdot \mathbf{U}(\mathbf{Y}; \mathbf{X}) = 0 \quad \text{and} \quad \mathbf{U}(\mathbf{Y}; \mathbf{0}) = \mathbf{0} \quad \text{for all } \mathbf{Y};$$

the function ϕ represents the solution of the differential inequality

$$(3.12) \quad T\psi(\mathbf{Y}; \mathbf{X}) = \mathbf{X} \cdot \nabla_x \phi(\mathbf{Y}; \mathbf{X}) \geq 0$$

and is given by

$$(3.13) \quad \phi(\mathbf{Y}; \mathbf{X}) = h(\mathbf{Y}) + \int_0^1 T\psi(\mathbf{Y}; \lambda \mathbf{X}) \frac{d\lambda}{\lambda},$$

where $h(\mathbf{Y})$ is an arbitrary, physically irrelevant function.

For our purposes, the possibility of the representation (3.10) of the vector \mathbf{I} and the existence of a twice continuously differentiable function ϕ , the dissipation potential, is important. With the properties of ϕ we conclude

$$(3.14) \quad \frac{\partial^2 \phi}{\partial X_\alpha \partial X_\beta} = \frac{\partial}{\partial X_\beta} (I_\alpha - U_\alpha) = \frac{\partial}{\partial X_\alpha} (I_\beta - U_\beta),$$

where the coordinates of the 9-dimensional vectors are indicated by Greek suffixes. The definitions (3.5) and (3.6) render explicitly for the relations (3.10)

$$(3.15) \quad \begin{aligned} S_i &= \frac{\partial \phi}{\partial E_i} + A_i, \\ -\frac{1}{T} q_i &= \frac{\partial \phi}{\partial T_{,i}} + C_i, \\ -{}_L H_i - \frac{\partial F}{\partial J_i} - \frac{1}{J_s^2} \varepsilon_{ijk} J_j R_k &= \frac{\partial \phi}{\partial \dot{J}_i} + G_i. \end{aligned}$$

The vectors \mathbf{A} , \mathbf{C} , \mathbf{G} are composed of the coordinates of the 9-dimensional vector \mathbf{U} and are (see Eq. (3.11)) restricted to

$$(3.16) \quad E_i A_i + T_{,i} C_i + \dot{J}_i G_i = 0.$$

They represent effects not connected with entropy production, e.g. the extraordinary Hall effect [7] through \mathbf{A} . In the following, we neglect such phenomena in our description and satisfy Eq. (3.16) by setting the vectors \mathbf{A} , \mathbf{C} , \mathbf{G} equal to zero. The relations (3.14) can be collected in the set

$$(3.17) \quad \begin{aligned} \frac{\partial S_j}{\partial E_i} &= \frac{\partial S_i}{\partial E_j}, & \frac{\partial q_j}{\partial T_{,i}} &= \frac{\partial q_i}{\partial T_{,j}}, \\ \varepsilon_{ijk} J_l \frac{\partial R_k}{\partial \dot{J}_i} &= \varepsilon_{ilk} J_l \frac{\partial R_k}{\partial \dot{J}_j}, & -\frac{1}{T} \frac{\partial q_i}{\partial E_j} &= \frac{\partial S_j}{\partial T_{,i}}, \\ -\frac{1}{J_s^2} \varepsilon_{ijk} J_l \frac{\partial R_k}{\partial E_j} &= \frac{\partial S_j}{\partial \dot{J}_i}, & \frac{1}{J_s^2} \varepsilon_{ijk} J_l \frac{\partial R_k}{\partial T_{,j}} &= \frac{1}{T} \frac{\partial q_j}{\partial \dot{J}_i}, \end{aligned}$$

where Eq. (3.1) has been used. We note that the last two relations are identically satisfied by the choice of the variables (3.1); the remaining equations represent nonlinear generalizations of Onsager-relations of classical thermodynamics.

In the following some simplifying assumptions are made. First, the dissipation potential is taken as being independent of exchange interactions, i.e. the polarisation gradient $J_{i,j}$. This is reasonable since, first, no derivatives with respect to $J_{i,j}$ occur; secondly, the damping moment \mathbf{R} is, in essence, caused by interaction of the spin continuum with conduction electrons and lattice; and, finally, the heat flux density results from lattice oscillations and conduction electrons and can therefore, like the electric current density, be considered as not being strongly affected by exchange forces.

Since the Hall effect has not been taken into account, we consequently neglect also other magneto-galvanic phenomena (magnetoresistive and magneto-thermoelectric effects) and assume the dissipation potential as being independent of magnetic polarisation, but not of the time rate $\dot{\mathbf{J}}$.

Since the drift velocity of electrons is much smaller than the speed of propagation of disturbances in polarisation, the damping of these disturbances can be considered as not being affected by electric current density and, as a consequence, by the electric field strength \mathbf{E} . Since the damping effect is a local one, we assumed in Eq. (3.1) independence of the temperature gradient. On the other hand, neglect of dynamic conduction phenomena led to the independence of $\dot{\mathbf{J}}$ in Eq. (3.1)_{4,5}.

Summing up, we write the dissipation potential in the form

$$(3.18) \quad \phi = \phi_1(T, T_{,i}, E_i) + \phi_2(T, \dot{J}_i).$$

Furthermore, we split Eq. (3.15)₃ with $\mathbf{G} = \mathbf{0}$ into the two relations

$$(3.19) \quad -{}_L H_i - \frac{\partial F}{\partial J_i} = 0 \quad \text{and} \quad -\frac{1}{J_s^2} \varepsilon_{ijk} J_j R_k = \frac{\partial \phi}{\partial \dot{J}_i},$$

since, as shown by Eqs. (2.7) and (3.1), the free energy density and the local anisotropy field strength are not dependent of $\dot{\mathbf{J}}$.

Let

$$(3.20) \quad F(J_i, J_{i,j}, T) \quad \text{and} \quad \phi_1(T, T_{,i}, E_i) + \phi_2(T, \dot{J}_i)$$

be given functions whose values represent the free energy density and energy dissipation, respectively, for special processes. The entropy density and the local anisotropy field strength are then determined (see Eqs. (3.2) and (3.19)) by

$$(3.21) \quad S = -\frac{\partial F}{\partial T}, \quad {}_L H_i = -\frac{\partial F}{\partial J_i}.$$

The exchange tensor is given by

$$(3.22) \quad t_{ij} = \frac{\partial F}{\partial J_{i,j}}$$

and obeys the conditions (2.10) and (2.12),

$$(3.23) \quad \varepsilon_{ijk} J_i J_{j,l} t_{kl} = 0, \quad J_i t_{ij} = 0.$$

From Eqs. (3.15) we obtain for the electric current density and heat flux density

$$(3.24) \quad S_i = \frac{\partial \phi_1}{\partial E_i} \quad \text{and} \quad q_i = -T \frac{\partial \phi_1}{\partial T_{,i}},$$

respectively, where the symmetry and cross relations (3.17)

$$(3.25) \quad \frac{\partial S_j}{\partial E_i} = \frac{\partial S_i}{\partial E_j}, \quad \frac{\partial q_j}{\partial T_{,i}} = \frac{\partial q_i}{\partial T_{,j}}, \quad \frac{\partial q_i}{\partial E_j} = -T \frac{\partial S_j}{\partial T_{,i}}$$

are valid, independent of special material symmetries. For the damping moment we get from Eq. (3.19) with Eqs. (2.10) and (3.17)

$$(3.26) \quad R_i = \varepsilon_{ijk} J_j \frac{\partial \phi_2}{\partial \dot{J}_k}, \quad J_k \frac{\partial}{\partial \dot{J}_k} \left(J_j \frac{\partial \phi_2}{\partial \dot{J}_i} - J_i \frac{\partial \phi_2}{\partial \dot{J}_j} \right) = 0.$$

Furthermore, we have the condition (3.3)

$$(3.27) \quad \psi T = E_i S_i - T_{,i} \frac{1}{T} q_i - \frac{1}{J_s^2} \varepsilon_{ijk} \dot{J}_i J_j R_k = E_i \frac{\partial \phi_1}{\partial E_i} + T_{,i} \frac{\partial \phi_1}{\partial T_{,i}} + \dot{J}_i \frac{\partial \phi_2}{\partial \dot{J}_i} \geq 0,$$

as a restriction on ϕ_1 and ϕ_2 .

The remaining part of the local energy balance (2.13),

$$(3.28) \quad q_{i,i} = -T \dot{S} + S_i E_i - \frac{1}{J_s^2} \varepsilon_{ijk} \dot{J}_i J_j R_k + r,$$

represents an extended form of the equation of heat conduction.

4. Special constitutive equations. Uniaxial ferromagnet

A simple expression for the dissipation potential is

$$(4.1) \quad \begin{aligned} \phi_1 &= \frac{1}{2} \sigma_{ij} E_i E_j + \beta_{ij} E_i T_{,j} + \frac{1}{2} \frac{1}{T} \lambda_{ij} T_{,i} T_{,j}, \\ \phi_2 &= \frac{1}{2\mu_0} \tau_{ij} \dot{J}_i \dot{J}_j, \end{aligned}$$

where the tensors σ_{ij} , λ_{ij} , τ_{ij} can, without loss of generality, be assumed to be symmetric and are in general as β_{ij} functions of the temperature. The condition (3.27) requires positive semi-definiteness of the form

$$\sigma_{ij} E_i E_j + 2\beta_{ij} E_i T_{,j} + \frac{1}{T} \lambda_{ij} T_{,i} T_{,j} + \frac{1}{\mu_0} \tau_{ij} \dot{J}_i \dot{J}_j = 2\phi$$

for arbitrary vectors E_i , $T_{,i}$, \dot{J}_i , so that the tensors σ_{ij} , λ_{ij} and τ_{ij} as well as

$$\left(\begin{array}{c|c} \sigma_{ij} & \beta_{ij} \\ \hline \beta_{ij} & \frac{1}{T} \lambda_{ij} \end{array} \right)$$

have to be positive semi-definite. From the relations (3.24) and (3.26) we obtain

$$(4.2) \quad \begin{aligned} S_i &= \sigma_{ij} E_j + \beta_{ij} T_{,j}, \\ q_i &= -\lambda_{ij} T_{,j} - \beta_{ij} T E_j, \\ R_i &= -\frac{1}{\mu_0} \varepsilon_{ijk} J_j \tau_{kl} \dot{J}_l; \end{aligned}$$

the relations (3.25) are satisfied by our choice and Eq. (3.26)₂ requires

$$(4.3) \quad (J_j \tau_{ik} - J_i \tau_{jk}) J_k = 0$$

as a condition for the damping tensor τ_{ij} . σ_{ij} represents the electric conductivity tensor for the vanishing temperature gradient and λ_{ij} the thermal conductivity tensor for the vanishing electric field. β_{ij} describes the Thomson effect [8].

For uniaxial anisotropy with the unit vector \mathbf{n} in the direction of the anisotropy axis, material tensors of second rank assume the canonical form [9]

$$(4.4) \quad A_{ij} = a_1 \delta_{ij} + a_2 n_i n_j$$

and we obtain

$$\begin{aligned} \sigma_{ij} &= \sigma(\delta_{ij} + s n_i n_j), \\ \lambda_{ij} &= \lambda(\delta_{ij} + l n_i n_j), \\ \beta_{ij} &= -\sigma \kappa(\delta_{ij} + b n_i n_j), \\ \tau_{ij} &= \tau(\delta_{ij} + r n_i n_j), \end{aligned}$$

with

$$(4.5) \quad \begin{aligned} \sigma \geq 0, \quad \lambda \geq 0, \quad \tau \geq 0, \quad \kappa^2 \sigma T \leq \lambda, \quad \kappa^2 \sigma T(1+b)^2 \leq \lambda(1+s)(1+l), \\ s \geq -1, \quad l \geq -1, \quad r \geq -1. \end{aligned}$$

$-T d\kappa/dT$ is the Thomson coefficient. In vector notation we have

$$(4.6) \quad \begin{aligned} \mathbf{S} &= \sigma[\mathbf{E} - \kappa \nabla T + \mathbf{n}(s \mathbf{n} \cdot \mathbf{E} - \kappa b \mathbf{n} \cdot \nabla T)], \\ \mathbf{q} &= -\lambda[\nabla T + l \mathbf{n}(\mathbf{n} \cdot \nabla T)] + \sigma \kappa T[\mathbf{E} + b \mathbf{n}(\mathbf{n} \cdot \mathbf{E})], \end{aligned}$$

and the condition (4.3), gives $\tau r(\mathbf{J} \times \mathbf{n})(\mathbf{J} \cdot \mathbf{n}) = 0$.

If the polarisation is not restricted to special directions with respect to the anisotropy axis, the damping is isotropic, $r = 0$, hence,

$$(4.7) \quad \mathbf{R} = \frac{\tau}{\mu_0} \mathbf{J} \times \dot{\mathbf{J}}.$$

Now, we turn to the free energy density function. Assuming that a uniaxial ferromagnet which exhibits spatially uniform and temporally constant polarisation along its anisotropy axis is subjected to a small perturbation

$$(4.8) \quad \mathbf{J} = J_0 \mathbf{n} + \mathbf{j},$$

where (see the relation (2.9))

$$J_0^2 = J_s^2, \quad \mathbf{j} \cdot \mathbf{n} = 0.$$

A possible expression for the free energy density for this special case is

$$(4.9) \quad F = F_0 + \frac{1}{2\mu_0} \beta j^2 + \frac{1}{2\mu_0} \alpha_{ij} j_{k,i} j_{k,j}.$$

$\beta j^2/2\mu_0$ with $\beta \geq 0$ represents the anisotropy energy [8] and $\alpha_{ij} j_{k,i} j_{k,j}/2\mu_0$ with $\alpha_{ij} = \alpha(\delta_{ij} + \rho n_i n_j)$ (see Eq. (4.4)), the exchange energy. Temperature variations connected with the perturbation are assumed to be small so that the coefficients can be regarded as con-

stants. Perturbations of the anisotropy-field strength and the exchange tensor follow from Eqs. (3.21), (3.22) as

$${}_L H_i = -\frac{1}{\mu_0} \beta j_i \quad \text{and} \quad t_{ij} = \frac{1}{\mu_0} \alpha_{jk} j_{i,k},$$

respectively; the conditions (3.23) are met, and the damping moment assumes the form

$$\mathbf{R} = \frac{\tau}{\mu_0} J_0 \mathbf{n} \times \dot{\mathbf{j}}.$$

If the magnetic field strength is taken as

$$\mathbf{H} = H_0 \mathbf{n} + \mathbf{h}, \quad H_0 = \text{const.},$$

the spin equation (2.4) can be written as

$$(4.10) \quad \frac{1}{\gamma} \dot{\mathbf{j}} = J_0 \mathbf{n} \times \mathbf{h}_{\text{eff}},$$

where we have introduced the effective magnetic field strength

$$(4.11) \quad \mu_0 (h_{\text{eff}})_i = \mu_0 h_i - \left(\beta + \frac{\mu_0 H_0}{J_0} \right) j_i + \alpha_{jl} j_{l,j} - \tau \dot{j}_i.$$

The relation (4.10) with Eq. (4.11) is the central equation for the investigation of perturbations in uniaxial ferromagnets. If these perturbations are represented by their Fourier components $\hat{\mathbf{j}}(\mathbf{k}, \omega)$ and $\hat{\mathbf{h}}(\mathbf{k}, \omega)$,

$$(4.12) \quad \mathbf{j}(\mathbf{x}, t) = \int \hat{\mathbf{j}}(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} d\mathbf{k} d\omega, \quad \mathbf{h}(\mathbf{x}, t) = \int \hat{\mathbf{h}}(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} d\mathbf{k} d\omega,$$

we get from the relation (4.10)

$$(4.13) \quad \hat{j}_i = \mu_0 \chi_{ij} \hat{h}_j$$

with the susceptibility tensor

$$(4.14) \quad \chi_{ij} = \frac{\gamma J_0}{\mu_0} \frac{(\Omega - i\omega\zeta)(\delta_{ij} - n_i n_j) + i\omega \varepsilon_{ijk} n_k}{(\Omega - i\omega\zeta)^2 - \omega^2}$$

and the abbreviations

$$(4.15) \quad \Omega = \frac{\gamma J_0}{\mu_0} \left(\beta + \frac{\mu_0 H_0}{J_0} + k_i \alpha_{ij} k_j \right), \quad \zeta = \frac{\gamma J_0}{\mu_0} \tau.$$

Equation (4.15) can be used immediately to describe such phenomena as ferromagnetic resonance and propagation of spin waves in uniaxial ferromagnets of the easy axis type.

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