

On the optimal control of the discretization problems for elastic bodies

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A DISCRETIZED elastic body is treated here as the continuous medium with constraints. A theory for such medium was formulated by Cz. WOŹNIAK. This approach enables us to obtain the basic system of equations in terms of the ordinary differential equations and estimate through the reaction forces a degree of accuracy of the solutions of the boundary value problems. The main aim of this paper is to solve the discretization control problem in such a manner that at the given number of finite elements and given norm the optimal net of division, i.e. a net of points for which the reaction forces reach minimum, is obtained.

Przedmiotem rozważań jest dyskretyzowane ciało sprężyste traktowane jako ośrodek ciągły z więzami, którego teorię sformułował Cz. WOŹNIAK. Podejście to umożliwiło otrzymanie podstawowego układu równań jako układu równań różniczkowych zwyczajnych oraz przeprowadzenie poprzez siły reakcji oceny stopnia dokładności rozwiązań problemów brzegowych. Zasadniczym elementem pracy jest rozwiązanie zagadnienia sterowania dyskretyzacją w ten sposób, aby przy danej liczbie elementów skończonych oraz normie uzyskać optymalną siatkę podziału, tj. siatkę punktów, przy której siły reakcji przyjmują minimum.

Предметом рассуждений является дискретизированное упругое тело, которое трактуется как сплошную среду со связями, теорию которой сформулировал Ч. Возняк. Этот подход дал возможность получить основную систему уравнений как систему обыкновенных дифференциальных уравнений, а также провести, через силы реакции, оценку степени точности решений краевых задач. Основным элементом работы является решение задачи управления дискретизацией таким образом, чтобы при заданном количестве конечных элементов и норме получить оптимальную сетку разбиения, т. е. сетку точек, при которой силы реакции принимают минимум.

THE subject of consideration is a discretized elastic body, the motion of which is described by the finite system of the unknown generalized displacements dependent solely on time, and determined by a prescribed set of points. This approach, leading out from theoretical mechanics has been recently known as a finite element method. In this paper a discretized body is treated as a material continuum with constraints. This enables us to reduce the basic system of equations to a system of ordinary differential equations and estimate, through the reaction forces, a degree of accuracy of the solutions of boundary-value problems. Assuming the reaction forces to be negligibly small in comparison to the external forces (mass and surface forces) acting on the body, it is possible to obtain a solution sufficiently close to the solution which could be obtained for a continuum without constraints.

The main aim of this paper is to solve the discretization control problem in such a way that at a given number of finite elements and given norm, the optimal net of division, i.e. a net of points at which the reaction forces reach a minimum, is obtained.

1. Introduction

Consider an elastic body B with particles \mathbf{X} and a fixed configuration κ . Assume the body B is discretized and is subjected to the action of constraints [1]. It means that the body may be divided into the finite number of the disjoint regions B_a , $a = 1, 2, \dots, l$ so called finite elements⁽¹⁾ or elements such that

$$\bar{B} = \bigcup_{a=1}^l \bar{B}_a.$$

It also means that the motion χ of each particle $\mathbf{X} \in B_a$ is the known function of the form:

$$(1.1) \quad \chi_a(\mathbf{X}, \tau) = \Phi_a(\mathbf{X}, \mathbf{q}(\tau)),$$

dependent on certain unknown variables $\mathbf{q}(\tau)$ called generalized deformations.

Assume also that the functions $\mathbf{q}(\tau)$ satisfy for every a , $d = d(a)$ the following conditions:

$$(1.2) \quad \gamma_a(\mathbf{q}(\tau)) = 0.$$

The conditions (1.2) may characterize the material and boundary conditions superposed on the functions $\mathbf{q}(\tau)$.

In the case when the body B is hiperelastic the constitutive equations have the form [1]

$$(1.3) \quad \mathbf{h} = - \sum_{a=1}^l \frac{\partial \varepsilon_a}{\partial \mathbf{q}},$$

where

$$(1.4) \quad \varepsilon_a = \int_{\kappa(B_a)} \varrho_a \sigma_a(\mathbf{X}, \nabla \chi_a) dV - \lambda \gamma_a.$$

The function $\sigma_a(\mathbf{X}, \nabla \chi_a)$ is the elastic energy of the element B_a , whereas λ are the Lagrange multipliers.

Introducing notations

$$(1.5) \quad \mathbf{f}(\tau, \mathbf{q}) = \sum_{a=1}^l \left[\int_{\kappa(B_a)} \varrho_a \mathbf{b}_a \frac{\partial \Phi_a}{\partial \mathbf{q}} dV + \int \mathbf{p}_a \frac{\partial \Phi_a}{\partial \mathbf{q}} dS \right],$$

$$k = \frac{1}{2} \dot{\mathbf{q}} \dot{\mathbf{q}} \sum_{a=1}^l \int_{\kappa(B_a)} \varrho_a \frac{\partial \Phi_a}{\partial \mathbf{q}} \frac{\partial \Phi_a}{\partial \mathbf{q}} dV,$$

where \mathbf{b}_a and \mathbf{p}_a are external forces and surface loading, respectively, the equations of motion of the discretized bodies with constraints may be written in the form

$$(1.6) \quad \mathbf{h} + \mathbf{f} = \frac{d}{d\tau} \frac{\partial k}{\partial \dot{\mathbf{q}}} - \frac{\partial k}{\partial \mathbf{q}}.$$

The equations of motion (1.6), the constitutive equations (1.3) together with the known initial conditions, constitute the initial problem for the functions $\mathbf{q}(\tau)$ and Lagrange multi-

⁽¹⁾ Here finite elements are understood in the same sense as in the well known finite element method.

pliers λ . After the evaluation of the functions \mathbf{q} and λ the motion of a continuum is determined from the Eqs. (1.1), whereas the state of stress is determined from the relations

$$(1.7) \quad \mathbf{T}_a(\mathbf{X}, \tau) = \int_{\sigma=0}^{\infty} (\mathbf{X}, \nabla \chi(\mathbf{X}, \tau - \sigma)).$$

The action of constraints (1.1) leads to the arising of the reaction forces \mathbf{r}_a ; \mathbf{s}_a , \mathbf{s}_{ab} , [1], where

$$(1.8) \quad \begin{aligned} \varrho_a \mathbf{r}_a &= \varrho_a \chi_a - \operatorname{div} \mathbf{T}_a - \varrho_a \mathbf{b}_a, & \mathbf{X} \in \kappa(B_a), & a = 1, 2, \dots, l, \\ \mathbf{s}_a &= \mathbf{p}_a - \mathbf{T}_a \mathbf{n}_a, & \mathbf{X} \in \partial \kappa(B_a) \cap \partial \kappa(B), \\ \mathbf{s}_{ab} &= \mathbf{T}_a \mathbf{n}_a + \mathbf{T}_b \mathbf{n}_b, & \mathbf{X} \in \partial \kappa(B_a) \cap \partial \kappa(B_b), \end{aligned}$$

and \mathbf{n}_a is an external normal versor to $\partial \kappa(B_a)$ and $\mathbf{n}_b = -\mathbf{n}_a$.

The surface reaction forces \mathbf{s}_a in the Eqs. (1.8) take into account the constraints connected with a discretization as well as with the supports of the boundary. In a case of boundary support the reaction forces of the boundary constraints will arise. Denoting these forces by \mathbf{s}_a^0 , the surface reaction forces of the constraints of discretization are differences $\mathbf{s}_a - \mathbf{s}_a^0$. Thus the reaction forces caused by a discretization are characterized by \mathbf{r}_a , $\mathbf{s}_a - \mathbf{s}_a^0$, \mathbf{s}_{ab} .

2. Linear constraints

Let $\{\mathbf{X}_{a\alpha}\}$, $\alpha = 1, 2, \dots, p$ denote a chosen series of points belonging to $\kappa(B_a) \cup \partial \kappa(B_a)$. The motion of the neighbourhood of the point $\mathbf{X}_{a\alpha}$ may be described by the function

$$(2.1) \quad \mathbf{y}_{a\alpha} = [\chi(\mathbf{X}_{a\alpha}, \tau), \chi_{,\gamma_1}(\mathbf{X}_{a\alpha}, \tau), \dots, \chi_{,\gamma_1 \dots \gamma_t}(\mathbf{X}_{a\alpha}, \tau)],$$

where $\gamma_r = 1, 2, 3$, $r = 1, 2, \dots, t$.

The functions $\mathbf{y}_{a\alpha} = \mathbf{y}_{a\alpha}(\tau)$ are assumed to be the generalized deformations $\mathbf{q}(\tau)$.

To obtain linear equations of motion it is assumed that:

$$(2.2) \quad \begin{aligned} \Phi_a(\mathbf{X}, \mathbf{u}_{a\alpha}) &= \sum_{\alpha} \mathbf{A}^{a\alpha} \mathbf{u}_{a\alpha}, \\ \Upsilon_a(\mathbf{u}_{a\alpha}) &= \sum_{\alpha} \mathbf{B}^{a\alpha} \mathbf{u}_{a\alpha}, \\ \sigma_a(\mathbf{u}_{a\alpha}) &= \frac{1}{2} \nabla \Phi_a \mathbf{C}_a \nabla \Phi_a^T, \end{aligned}$$

where $\mathbf{u}_{a\alpha}(\tau) = \mathbf{y}_{a\alpha}(\tau) - \mathbf{y}_{a\alpha}(\tau_0)$.

The Eqs. (2.2) imply that from the relations (1.1) and (1.2) only these equations are taken into consideration which may be approximated by the linear homogeneous functions.

Using the Eqs. (2.2), the equations of motion (1.6) may be written in the form:

$$(2.3) \quad \mathbf{D}^{\nu} \mathbf{u}_j + \mathbf{E}^i \lambda + \mathbf{f}^i = \frac{d}{d\tau} (\mathbf{a}^{\nu} \dot{\mathbf{u}}_j),$$

where the suffix i runs the number $1, 2, \dots, N$ and

$$\{1, 2, \dots, N\} = \{a_\alpha; a = 1, 2, \dots, l, \alpha = 1, 2, \dots, p\}$$

and

$$(2.4) \quad \begin{aligned} \mathbf{D}^{ij} &= - \sum_{a=1}^l \int_{\kappa(B_a)} \varrho_a \nabla \mathbf{A}^{a_\alpha} \mathbf{C}_a \nabla \mathbf{A}_{a_\beta}^{a_\alpha} dV, \\ \mathbf{E}^i &= \sum_{a=1}^l \mathbf{B}^{a_\alpha}, \\ \mathbf{a}^{ij} &= \sum_{a=1}^l \int_{\kappa(B_a)} \varrho_a \mathbf{A}^{a_\alpha} \mathbf{A}^{a_\beta} dV. \end{aligned}$$

Here summation \sum^* runs through all values of a for which

$$i = a_\alpha, \quad j = a_\beta, \quad \alpha, \beta = 1, 2, \dots, p.$$

The suffix i introduced here for numerical convenience enumerates all points \mathbf{X}_{a_α} through \mathbf{X}_i . The set of points \mathbf{X}_i is called a net of points for a discretized body B .

The Eqs. (2.3) and (2.2) form a set of the linear displacement equations of motion for the discretized bodies. Mathematically they constitute a system of linear second-order ordinary differential equations. In the case when \mathbf{D}^{ij} , \mathbf{E}^i do not depend on time we have a system with constant coefficients.

For the quasi-static problems the right-hand sides of the Eqs. (2.3) are equal to zero. Then the relations (2.3) and (2.2) form a nonhomogeneous algebraic system of equations for determining of the unknown \mathbf{u}_i and λ .

3. Optimal control

Although each body may be discretized, i.e. "divided on the finite elements", the results obtained for discretized bodies describe the system under consideration (in fact without constraints) within the reaction forces.

After evaluation of the generalized deformations $\mathbf{q}(\tau)$ from the equations of motion (1.6) and the constitutive equations (1.7), the total reaction forces may easily be determined with the help of formulae (1.8).

Defining the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, in a space of external and surface forces ($\mathbf{b}_a, \mathbf{p}_a$) as well as the reaction forces the solution obtained in a process of discretization $\{\mathbf{X}_i\}$, $i = 1, 2, \dots, N$ is considered to be sufficiently close to the solution which could be obtained for a body without constraints if for every a the following estimating condition

$$(3.1) \quad (\|\mathbf{r}_a\|_1 + \|\mathbf{S}_a\|_2 + \|\sum_{b=1}^l \mathbf{s}_{ab}\|_2 = \epsilon (\|\mathbf{b}_a\|_1 + \|\mathbf{p}_a + \mathbf{s}_a^0\|_2),$$

is satisfied. Here $\mathbf{S}_a = \mathbf{s}_a - \mathbf{s}_a^0$ and ϵ is a given positive number sufficiently small with respect to 1. For practical use $\epsilon = 0,05$ may be assumed, since with such accuracy the ex-

ternal forces acting on a body are determined. This condition (3.1) says that the external and surface reaction forces should constitute the system of forces negligibly small in comparison to the system of external forces acting on a body. Starting with evaluation of the optimal net of division let us denote the set of points \mathbf{X} by Ω_0

$$\Omega_0 = \{\mathbf{X}; \mathbf{X} \in \bigcup_{\alpha=1}^l \partial\kappa(B_\alpha) \cup \partial\kappa(B)\}.$$

The set $\Omega_0 \subset E^\alpha$, $\alpha = 1, 2, 3$ is composed of the net points $\{\mathbf{X}_i\}$ when $\alpha = 1$, net points and arcs $\{(\mathbf{X}_i, \mathbf{X}_j)\}$ when $\alpha = 2$ and net points, arcs and sheets $\{(\mathbf{X}_i, \mathbf{X}_j, \dots, \mathbf{X}_h)\}$ when $\alpha = 3$.

A unique decomposition of the set Ω_0 on the nets of points, arcs and sheets results from the manner of discretization (since the finite elements are disks). The set Ω_0 will be called further α -dimensional net of the discretized body, where α corresponds to the case of one, two- and three-dimensional body.

Furthermore let h be a homeomorphism $h: \kappa(\overline{B}) \rightarrow \kappa(\overline{B})$. As it is easy to verify by using the elementary notions of topology, the homeomorphisms form a group of transformations which we will denote by \mathcal{H} .

In particular the homeomorphisms h , which are identities on the boundary, form a subgroup \mathcal{H} .

Let us define now the homeomorphisms $g \in G$ in the following way

$$g: \Omega_0 \rightarrow \Omega \subset \kappa(\overline{B}) \wedge g(\mathbf{X}) = h(\mathbf{X}), \mathbf{X} \in \Omega_0.$$

The homeomorphism g is then a transaction of a homeomorphism h to the net Ω_0 . The set Ω is also an α -dimensional net of the body B . This fact results immediately from the properties of the homeomorphisms. And on the contrary, each net Ω describing the discretization of the body B by means of the same number of finite elements B_α is the image of the net Ω_0 , i.e. there exists such a homeomorphism that $g^{-1}(\Omega) = \Omega_0$. It is evident that $\bigcup_{g \in G} g(\Omega_0) = \kappa(\overline{B})$.

The motion of a discretized body in every moment of time τ and for each net Ω is described by the generalized displacements \mathbf{u}_i , determined in points $\mathbf{y}_i = g(\mathbf{x}_i)$, $i = 1, 2, \dots, N$ and $g(\Omega_0) = \Omega$.

The system of differential equations (2.3) with initial conditions

$$(3.2) \quad \mathbf{u}_i(\tau_0) = \mathring{\mathbf{u}}_i, \quad \dot{\mathbf{u}}_i(\tau_0) = \mathring{\mathbf{V}}_i$$

serves for evaluation of the functions \mathbf{u}_i .

The generalized displacements \mathbf{u}_i at the fixed τ and fixed net Ω belong to the Euclidean space E^{4Nm} . We shall call them the vectors of state or phase vectors and the space E^{4Nm} — the space of state.

A set of nets Ω , where $\Omega = g(\Omega_0)$, $g \in G$ will be denoted by Θ and called a domain of control. In many particular cases the nets Ω will belong to a certain subset $\overline{\Theta} \subset \Theta$. A specification of this subset in the set Θ depends on the conditions superposed on the

homeomorphism g (i.e. the piece-wise linear conditions) as well as on a domain of determinancy g . In fact, the control of the whole net Ω is not always needed but only a part.

To every solution \mathbf{u}_i with boundary conditions (3.2) the number V may be attributed in the following way:

$$(3.3) \quad V = V(\mathbf{u}_i, \Omega) = \sum_{a=1}^l (\|\mathbf{r}_a(\mathbf{u}_i, \Omega)\|_1 + \|\mathbf{S}_a(\mathbf{u}_i, \Omega)\|_2) + \|\sum_{b=1}^l \mathbf{s}_{ab}(\mathbf{u}_i, \Omega)\|_2,$$

where $\Omega \in \Theta$.

We say that the control of the net Ω^* is optimal if the solution \mathbf{u}_i^* of the state equations (2.3) at the initial conditions (3.2) and corresponding to this net, satisfies for $\Omega \in \Theta$ the following relation:

$$(3.4) \quad V(\mathbf{u}_i^*, \Omega^*) \leq V(\mathbf{u}_i, \Omega).$$

Thus by the finding of an optimal net of division of the body B we shall understand the finding of such element of the set Θ for which the functional (3.4) takes the smallest value.

4. A method of evaluation of the optimal discretization

A basic problem in the solution of the optimal control of the discretization problem is to determine the set of controls $\bar{\Theta}$, frequently called admissible. The difficulty in finding an optimal discretization (optimal net) depends in a great measure on this set.

Let $\bar{\Theta} \subset \Theta$ be a set of representatives of the following equivalency relation:

$$(4.1) \quad (\Omega_1 \approx \Omega_2) \Leftrightarrow [g_1(\mathbf{X}_i) = g_2(\mathbf{X}_i) \wedge g_1(\Omega_0) = \Omega_1, g_2(\Omega_0) = \Omega_2],$$

where $\Omega_1, \Omega_2 \in \Theta$.

The relation (4.1) is reflexive symmetric and transitive, it is then the equivalency relation.

For an abstraction class $\|\Omega\|$ of the equivalence relation (4.1) it is convenient to choose the representative $\bar{\Omega} \in \Theta$ in following way:

Let $\Omega = g(\Omega_0)$ and the images of arcs $(\mathbf{X}_i, \mathbf{X}_j)$ at g be segments and the images of sheets $(\mathbf{X}_i, \mathbf{X}_j, \mathbf{X}_r)$ be polygons.

In particular when the discretization Ω_0 is a triangulation of the body B by simplexes (in a two-dimensional case it takes place in a division into the triangular finite elements and in the spatial case into tetrahedral elements), then all triangulations of the body B arise from the triangulation Ω_0 form a set of representatives $\bar{\Theta}$.

Elements of the set Θ may be assigned mutually and uniquely to certain subsets in the Euclidean space.

Since the abstraction class $\|\Omega\|$ determines uniquely the net of points $\{\mathbf{Y}_i\}$, $i = 1, 2, \dots, N$, and *vice versa* a net of points $\{\mathbf{Y}_i\}$, according to the Eq. (4.1), determines uniquely $\|\Omega\|$, then the point $\mathbf{y} \in E^{nN}$ assigned to $\|\Omega\|$, where $\alpha = 1, 2, 3$ and $\mathbf{y} = (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_N)$.

Obviously, for the fixed i , $\mathbf{Y}_i \in \overline{\kappa(B)}$, so then the point \mathbf{y} in the Euclidean space E^{aN} , to which a certain element θ is assigned, belongs to N -multiple Cartesian product $\overline{\kappa(B)}$ from which the set

$$M = \{ \mathbf{y} : \mathbf{y} \in \overline{\kappa(B)} \times \dots \times \overline{\kappa(B)}, \bigvee_{i \neq j} \mathbf{y}_{\alpha i - a} = \mathbf{y}_{\alpha j - a}, a = 0, 1, \dots, \alpha - 1 \}$$

is removed. In this case a control domain may be identified with the set $S = \overline{\kappa(B)} \times \dots \times \overline{\kappa(B)} - M$.

In many boundary-value problems it is assumed that the homeomorphisms g are identities on $\partial\kappa(B)$. Let $\mathbf{X}_t \in \kappa(\partial B)$, $t = 1, 2, \dots, N_0$ and $N_0 \leq N$. Then a dimension of the domain of S is equal to $\alpha(N - N_0)$, and S is an open set (a sum of disjoint regions).

A functional (3.3) after taking into account the Eq. (3.1) may be rewritten in the form.

$$(4.2) \quad V = \int_{\tau_0}^{\tau_1} \sum_{a=1}^l \{ \| -\operatorname{div} \mathbf{T}_a(\mathbf{X}, \mathbf{Y}_i, \mathbf{u}_i) - \varrho_a(\mathbf{X}) \mathbf{b}_a(\mathbf{X}, \tau) + \varrho_a(\mathbf{X}) \ddot{\Phi}(\mathbf{X}, \mathbf{Y}_i, \mathbf{u}_i) \|_1 + \| \mathbf{T}_a(\mathbf{X}, \mathbf{Y}_i, \mathbf{u}_i) \mathbf{n}_a(\mathbf{X}, \mathbf{Y}_i) - \mathbf{p}_a(\mathbf{X}) - \mathbf{s}_a^0(\mathbf{X}) \|_2 + \| \sum_{b=1}^l \mathbf{T}_b(\mathbf{X}, \mathbf{Y}_i, \mathbf{u}_i) \mathbf{n}_b(\mathbf{X}, \mathbf{Y}_i) + \mathbf{T}_a(\mathbf{X}, \mathbf{Y}_i, \mathbf{u}_i) \mathbf{n}_a(\mathbf{X}, \mathbf{Y}_i) \|_2 \} d\tau,$$

where

$$\mathbf{T}_a(\mathbf{X}, \mathbf{Y}_i, \mathbf{u}_i) = \varrho_a(\mathbf{X}) \frac{\partial \sigma(\mathbf{X}, \nabla \Phi_a(\mathbf{X}, \mathbf{Y}_i, \mathbf{u}_i)}{\partial \nabla \Phi_a(\mathbf{X}, \mathbf{Y}_i, \mathbf{u}_i)}.$$

In a static case the integration with respect to time in the Eq. (4.2) will not occur. As an example of a functional of purpose (4.2) let

$$(4.3) \quad V = \int_{\tau_0}^{\tau_1} \sum_{a=1}^l \left[l_a \int_{\kappa(B_a)} | -\operatorname{div} \mathbf{T}_a - \varrho_a \mathbf{b}_a + \varrho_a \ddot{\Phi}_a |^2 dV + \int_{\partial\kappa(B_a) \sim \partial\kappa(B)} | \mathbf{T}_a \mathbf{n}_a - \mathbf{p}_a - \mathbf{s}_a^0 |^2 dS + \sum_{b=1}^l \int_{\kappa(\partial B_a) \sim \partial\kappa(B_b)} | \mathbf{T}_a \mathbf{n}_a + \mathbf{T}_b \mathbf{n}_b |^2 dS \right] d\tau$$

where the weight l_a is a quantity characterizing the element B_a (for example the diameter of B_a).

Let now $\mathbf{u}_i(\mathbf{Y}_i, \tau)$ be the solutions of the system (2.3) with initial conditions (3.2) where $\mathbf{Y}_i \in S$. Introducing \mathbf{u}_i to the functional (4.2), and performing corresponding operations such as integration and tensor multiplication, we obtain V as the function of the variables \mathbf{y} .

A problem of determining of the extremum of the multivariable function is easy to solve. Thus it is seen that, in the case when a set of admissible controls is a set of the equivalent nets with respect to the relation (4.1), the optimal control problem may be reduced to the significantly simpler problem of seeking extremum of the multivariable functions.

It may happen, however, that there is no point $\mathbf{Y}_i \in \operatorname{int} S$ in which the sufficient condition of the existence of extremum is fulfilled.

We must then look for the smallest value on the boundary S . In the case of a closed set S the existence of the smallest value results from the Weierstrass theorem.

In the case when S is an open set (for example at the fixed net points on $\partial\kappa(B)$) and the extremal points in S do not exist, the behaviour of the function V must be examined. Then from the analysis of this behaviour we must conclude how to change the distribution of points $Y_i \in \partial\kappa(B)$.

5. Example of the optimal control

Consider a plate layer loaded as shown in Fig. 1. We divide the section of the layer on four finite elements AEC , ABE , BED , ECD , where $A = (-1,0)$, $B = (1,0)$, $C = (-1,1)$, $D = (1,1)$, $E = (0, a)$. A , B , C , D and E are the net points. Because of the

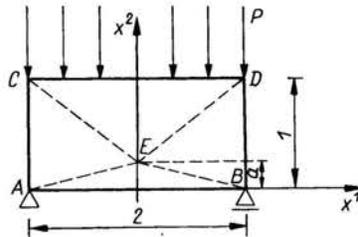


FIG. 1.

symmetry of the problem we consider only these homeomorphisms which are identities on the boundary and transform the point E into set

$$\{x^k, x^1 = 0, \quad 0 < x^2 < 1\}.$$

Thus $N = 5$, $N_0 = 4$, $s = (0, 1) \subset E^1$.

The equations of constraints (2.2) after taking into account the approach described in [2] take the form

$$(5.1) \quad \begin{aligned} \Phi_1^k &= [(a - (1-a)x^1 - x^2)u_A^k + (1+x^1)u_E^k + (-a - ax^1 + x^2)u_C^k], & \text{for } x^k \in \Delta AEC = B_1, \\ \Phi_2^k &= \frac{1}{2a} [(a - ax^1 - x^2)u_A^k + (a + ax^1 - x^2)u_B^k + 2x^2 u_E^k], & \text{for } x^k \in \Delta ABE = B_2, \\ \Phi_3^k &= \frac{1}{2(1-a)} [(-a + (a-1)x^1 + x^2)u_C^k + (2-2x^2)u_E^k + (-a + (1-a)x^1 + x^2)u_D^k], & \text{for } x^k \in \Delta CED = B_3, \\ \Phi_4^k &= -[(-a - (1-a)x^1 + x^2)u_B^k + (-1+x^1)u_E^k + (a - ax^1 - x^2)u_D^k], & \text{for } x^k \in \Delta DEB = B_4. \end{aligned}$$

Computing next the function of the elastic energy ($\lambda = 1$, $\mu = 1/2$) and solving the system of the Eqs. (2.3) we obtain

$$(5.2) \quad \begin{aligned} u_A^1 &= u_A^2 = u_B^1 = u_B^2 = u_E^1 = 0, \\ u_C^1 &= -u_D^1 = -\frac{1}{6}, \quad u_C^2 = u_D^2 = -\frac{1}{3}, \\ u_E^2 &= au_C^2. \end{aligned}$$

Substituting the Eq. (5.2) into the Eq. (5.1) we have:

$$\begin{aligned}\Phi_1^1 &= \frac{1}{6}(a+ax^1-x^2), & \Phi_2^1 &= 0, & \Phi_3^1 &= \frac{x^1}{6}, & \Phi_4^1 &= \frac{1}{6}(a-ax^1-x^2), \\ \Phi_\alpha^2 &= -\frac{x^2}{3}, & \alpha &= 1, 2, 3, 4.\end{aligned}$$

Computing the reaction forces s_a, s_{ab} from the relations (3.1) and substituting them into the Eq. (4.3) we get

$$(5.3) \quad V(a) = \frac{1}{144} \left[4a^2 - 8a + 85 + \frac{1}{(a^2+1)^{3/2}} (16a^4 - 64a^3 + 80a^2 - 111a + 265) + \frac{1}{(a^2-2a+2)^{3/2}} (16a^4 - 32a^3 + 233a^2 - 398a + 294) \right].$$

After evaluating $V'(a)$ we obtain $V'(a) > 0$ for $0 \leq a \leq 1$. Thus it is seen that in the interval $\langle 0, 1 \rangle$ the function (5.3) is decreasing and takes the smallest value in the point $a = 1$ (Fig. 2).

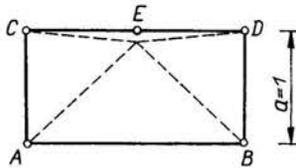


FIG. 2.

The example presented here has a test character. However, the analogical procedure may also be applied to the more combined problems of optimization of the division of a body on the finite elements.

References

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