

Stability of non-parallel flows

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THE METHOD of multiple scales is used to obtain a set of consistent equations governing the linear stability of slightly non-parallel, incompressible, steady flows. The numerical procedure for obtaining the solution of the non-parallel problem is outlined. The complete solution contains the solution of the Orr-Sommerfeld problem as the first approximation, the distortion of the Orr-Sommerfeld eigenfunctions, and the local perturbation and streamwise variation in the wave-number and spatial growth rate when the frequency of the disturbance and the Reynolds number of the primary flow are fixed.

Dla otrzymania układu równań rządzących liniową statecznością słabo nierównoległych, nieściśliwych, ustalonych przepływów zastosowano metodę wielu skal. Omówiono procedurę numeryczną stosowaną przy rozwiązywaniu zagadnienia nierównoległego przepływu. Pełne rozwiązanie zawiera jako pierwsze przybliżenie rozwiązanie zagadnienia Orra-Sommerfelda, opisuje zmianę funkcji własnych tego zagadnienia jak również uwzględnia zaburzenia lokalne oraz zmienność liczby falowej i wzrost przestrzenny, występujące wzdłuż linii prądu przy założeniu, że częstość zaburzeń i liczba Reynoldsa przepływu pierwotnego są stałe.

Для получения системы уравнений, описывающих линейную устойчивость слабо непараллельных, несжимаемых, установившихся течений, применен метод многократных шкал. Обсуждена численная процедура применяемая при решении проблемы непараллельного течения. Полное решение содержит, как первое приближение, решение задачи для собственных функций Орра-Зоммерфельда, как тоже учитывает локальные возмущения, а также переменность волнового числа и пространственный рост, выступающий вдоль линий тока, при предположении, что частота возмущений и число Рейнольдса первичного течения постоянны.

1. Introduction

THE LINEAR stability of slightly non-parallel flows has received considerable attention in recent years. Some attempts to account for the non-parallelism of the primary flow involved the retention of the normal component of velocity and some of the streamwise derivatives of the primary flow in the stability equation [see, e.g., BOEHMAN (1971), BARRY and ROSS (1970), and MORKOVIN (1969)]. The disturbance was given the form of a streamwise-traveling wave having an amplitude which is a function of the transverse coordinate. Streamwise variations in the wave-number, spatial growth rate, and amplitude were neglected. Neglecting these variations is inconsistent, however, because they are of the same order as some of the terms retained. The same kind of inconsistency is also associated with attempts to determine the effects of blowing and suction [see., e.g., CHEN and HUANG (1972), KOBAYASHI (1972), and CHEN, SPARROW, and TSOU (1971)], and of vorticity in the outer flow [WERLE MOOK, and TANG (1973)].

This inconsistency was removed by LING and REYNOLDS (1973), who correctly accounted for some of the streamwise variations through a perturbation about the parallelflow solution. The perturbation, which was effected by assuming expansions for all the

quantities, was structured to provide local perturbations in frequency, temporal growth rate, and Reynolds number for a given wave-number; however, the results also contain the local distortion of the eigenfunction as well as the streamwise variation of the wave-number. The spatial growth rate was taken to be zero.

In the present paper we develop an alternate approach which is based on the method of multiple scales [NAYFEH (1973)]. The present approach is structured to provide perturbations in the wave-number and spatial growth rate for fixed frequency and Reynolds number; it also yields the local distortion and streamwise variation of the eigenfunction as well as the streamwise variation in wave-number and spatial growth rate. The numerical procedure is outlined.

2. Problem formulation

We take the streamfunction of the disturbed flow $\hat{\psi}$ to be of the form

$$(2.1) \quad \hat{\psi}(x, y, t) = \Psi(x, y) + \psi(x, y, t),$$

where Ψ is the streamfunction of the steady, primary flow, which by itself satisfies the Navier-Stokes equation and the appropriate boundary conditions and is presumed to be known, and ψ is the streamfunction of the disturbance. Substituting Eq. (2.1) into the Navier-Stokes equation, neglecting non-linear terms in ψ , and introducing non-dimensional variables, we find that ψ is governed by

$$(2.2) \quad \frac{\partial}{\partial t}(\nabla^2\psi) + \frac{\partial\Psi}{\partial y} \frac{\partial}{\partial x}(\nabla^2\psi) + \frac{\partial}{\partial x}(\nabla^2\Psi) \frac{\partial\psi}{\partial y} - \frac{\partial\Psi}{\partial x} \frac{\partial}{\partial y}(\nabla^2\psi) - \frac{\partial}{\partial y}(\nabla^2\Psi) \frac{\partial\psi}{\partial x} = \frac{1}{R} \nabla^4\psi,$$

where R is the Reynolds number based on some convenient length. Equation (2.2) is valid for any two-dimensional primary flow and disturbance. In addition, ψ and $\partial\psi/\partial y$ must be zero along a rigid wall, and if the transverse dimension of the flow field is infinite, ψ must decay as the distance from the region where viscous effects are important increases.

When consideration is restricted to primary flows which are nearly parallel, it is convenient to introduce an additional independent variable in the streamwise direction. We put

$$(2.3)_1 \quad x_1 = \varepsilon x,$$

where ε is some measure of the non-parallelism of the primary flow; $\varepsilon = 0$ for truly parallel flows (ε and R may be related). x and x_1 are the so-called fast and slow scales, respectively. We assume that the streamfunction of the primary flow has the form

$$(2.3)_2 \quad \Psi = \Psi(x_1, y),$$

where y is the transverse coordinate. It follows that

$$(2.3)_3 \quad \frac{\partial\Psi}{\partial y} = U = U(x_1, y)$$

and

$$(2.3)_4 \quad -\frac{\partial\Psi}{\partial x} = -\varepsilon \frac{\partial\Psi}{\partial x_1} = \varepsilon V(x_1, y).$$

Because the coefficients in Eq. (2.2) vary slowly in the streamwise direction and are independent of time (they are functions of x_1 and y only), the disturbance streamfunction can be given the form

$$(2.3)_5 \quad \psi = [\phi_0(x_1, y) + \varepsilon\phi_1(x_1, y) + \dots] \exp(i\theta),$$

where

$$(2.3)_6 \quad \frac{\partial\theta}{\partial x} = k(x_1)$$

and

$$(2.3)_7 \quad \frac{\partial\theta}{\partial t} = -\omega \quad \text{a constant.}$$

Hence, the fast scale is used to describe the relatively rapid, streamwise variation of the traveling-wave disturbance, and the slow scale is used to describe the relatively slow variation of the primary flow and the wavenumber, spatial growth rate, and amplitude of the disturbance.

Substituting Eqs. (2.3) into Eq. (2.2) and equating coefficients of like powers of ε , we find that ϕ_0 and ϕ_1 are governed by

$$(2.4) \quad L(\phi_0) = \{(D^2 - k^2)^2 - iR[(kU - \omega)(D^2 - k^2) - kD^2U]\} \phi_0 = 0$$

and

$$(2.5) \quad L(\phi_1) = H,$$

where

$$D = \frac{\partial}{\partial y},$$

$$H = R \left[a_1 \frac{\partial\phi_0}{\partial x_1} + a_2 D^2 \left(\frac{\partial\phi_0}{\partial x_1} \right) + a_3 D\phi_0 + VD^3\phi_0 + \left(a_4 \phi_0 - \frac{i2}{R} D^2\phi_0 \right) \frac{dk}{dx_1} \right],$$

$$a_1 = -3Uk^2 - D^2U + 2k\omega + \frac{i4k^3}{R}, \quad a_2 = -\frac{i4k}{R} - U, \quad a_3 = -D^2V - k^2V,$$

$$a_4 = \omega - 3kU + \frac{i6k^2}{R}.$$

The boundary conditions for ϕ_0 and ϕ_1 may be chosen from the following:
 along a solid wall

$$\phi_0 = \phi_1 = D\phi_0 = D\phi_1 = 0,$$

along a plane of symmetry

$$D\phi_0 = D\phi_1 = D^3\phi_0 = D^3\phi_1 = 0,$$

along a plane of antisymmetry

$$\phi_0 = \phi_1 = D^2\phi_0 = D^2\phi_1 = 0.$$

If the transverse dimension of the flow field is infinite,

$$\phi_0 \text{ and } \phi_1 \rightarrow 0$$

as the distance from the viscous region increases.

The eigenvalue problem defined by Eq. (2.4) and the appropriate boundary conditions is the familiar Orr-Sommerfeld problem for parallel flows. Here, R and ω are considered (real) parameters and $k(x_1; R, \omega)$ is the (complex) eigenvalue. With this approach, consideration is given to the effect of the non-parallelism on the value of k for given R and ω . ϕ_0 may be written in the form

$$(2.6) \quad \phi_0 = A(x_1)\eta(x_1, y),$$

where η is the eigenfunction. At this point the amplitude function A is unknown; it is determined to within a multiple at the next level of approximation. If the flow is considered parallel, then A would be considered constant and Eq. (2.6) would be the complete solution.

In order for Eq. (2.5) to have a solution that satisfies the boundary conditions, H must satisfy the solvability condition:

$$(2.7) \quad \int_{y_2}^{y_1} H\phi^* dy = 0,$$

where y_1 and y_2 are on opposite edges of the flow field and ϕ^* satisfies

$$(2.8) \quad (D^2 - k^2)^2 \phi^* - iR[(kU - \omega)(D^2 - k^2)\phi^* + 2kDUD\phi^*] = 0,$$

the adjoint equation, and the same boundary conditions as ϕ_1 .

Equation (7) leads to

$$\frac{dA}{dx_1} - \alpha(x_1)A = 0,$$

and hence

$$A = C \exp \int \alpha(x_1) dx_1,$$

where

$$(2.9) \quad \alpha(x_1) = \frac{-\int_{y_1}^{y_2} \left[a_1 \frac{\partial \eta}{\partial x_1} + a_2 D^2 \left(\frac{\partial \eta}{\partial x_1} \right) + a_3 D\eta + VD^3\eta + \left(a_4 \eta - \frac{i2}{R} D^2 \eta \right) \frac{dk}{dx_1} \right] \phi^* dy}{\int_{y_1}^{y_2} (a_1 \eta + a_2 D^2 \eta) \phi^* dy}.$$

α gives the desired correction to k ; however, to evaluate α , one must first have η , $\partial\eta/\partial x_1$ and dk/dx_1 .

The following is needed to determine $\partial\eta/\partial x_1$ and dk/dx_1 . Because $L(\zeta) = 0$, differentiating leads to

$$(2.10) \quad L(\zeta) = R \frac{dk}{dx_1} \left\{ i[U(D^2 - 3k^2) - D^2 U + 2k\omega] + \frac{4k}{R} (D^2 - k^2) \right\} \eta + ik \left[\frac{\partial U}{\partial x_1} (D^2 - k^2) - D^2 \left(\frac{\partial U}{\partial x_1} \right) \right] \eta,$$

where $\zeta = \partial\eta/\partial x_1$ and the boundary conditions on ζ are the same as those for η . Applying the solvability condition to Eq. (2.10) yields

$$(2.11) \quad \frac{dk}{dx_1} = \frac{-ik \int_{y_1}^{y_2} \phi^* \left[\frac{\partial U}{\partial x_1} (D^2 - k^2) - D^2 \left(\frac{\partial U}{\partial x_1} \right) \right] \phi_0 dy}{\int_{y_1}^{y_2} \phi^* \left\{ i[U(D^2 - 3k^2) - D^2 U + 2k\omega] + \frac{4k}{R} (D^2 - k^2) \right\} \phi_0 dy}.$$

3. Method of solution for a given Ψ

First, we choose ω and R and then determine k and ϕ_0 from the Orr-Sommerfeld problem [Eq. (2.4) and the appropriate boundary conditions], using either the procedure described by LING and REYNOLDS or the one given by MOOK (1972). Next, we determine ϕ^* using Eq. (2.8). [One can vary this procedure, finding k and ϕ^* from Eq. (2.8) and the appropriate boundary conditions and then ϕ_0 from Eq. (2.4)]. dk/dx_1 is found from Eq. (2.11) and then Eq. (2.10) is integrated to yield ζ and consequently $D^2\zeta$. It should be noted that Eq. (2.11) does not insure a unique solution to Eq. (2.10). In fact, if ζ_p is a solution to Eq. (2.10), so is $\zeta_p + C\phi_0$, where C is arbitrary. We assume that ϕ_0 term should be eliminated and, following Ling and Reynolds, effect the elimination by subtracting directly the contribution of the homogeneous solution.

α is determined from Eq. (2.9). The corrected wave-number, which includes the effects of the nonparallelism, is

$$K_r = k_r + i\epsilon\alpha_i,$$

and the corrected spatial growth rate is

$$K_i = -k_i + \epsilon\alpha_r.$$

Finally, Eq. (2.5) is integrated to give ϕ_1 .

4. Summary

The linear stability of slightly non-parallel flows is analysed by perturbing about the parallel-flow solution. The method of multiple scales is used to effect this perturbation, which is structured to provide the changes in k (wave-number and spatial growth rate) for given ω (frequency of the disturbance) and R (Reynolds number of the primary flow). With the present approach, the solution of the Orr-Sommerfeld problem emerges as the first approximation. Accounting for the non-parallelism requires the solution of a sequence of problems.

The complete solution for the non-parallel problem contains the solution of the Orr-Sommerfeld problem (ϕ_0, k), the streamwise variation in k (dk/dx_1), the local perturbation in $k(\alpha)$, and the distortion of the eigenfunction of the Orr-Sommerfeld problem due to non-parallel effects (ϕ_1) for given ω and R .

Neutral stability curves can be constructed by solving the complete problem for different values of ω and for each ω determining the R for which $-k_i + \epsilon\alpha_r$ is zero by iteration.

For the Blasius boundary layer $\psi = (x_1)^{\frac{1}{2}} f[y(x_1)^{-\frac{1}{2}}]$, where $x_1 = \varepsilon x = x/R$ and $R = U_\infty \delta_1/\nu$ the Reynolds number based on displacement thickness. In this case, the calculations show a minimum Reynolds number of 396 and a maximum frequency $F = \omega\nu/U_\infty^2 = 400 \times 10^{-6}$ compared with $R = 520$ and $F = 250 \times 10^{-6}$ for the parallel theory. The non-parallel results compare favorably with the experimental results.

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