

The structure and decay of trailing vortices

P. G. SAFFMAN (PASADENA)

THE ROLL up of the trailing vortex sheet behind a high aspect ratio wing is examined. The results are used to infer the initial structure of the trailing vortices. The decay of the trailing vortices is considered. The existence of axial velocities towards the wing is explained. The effect of the internal structure on the growth rates of the mutual instability of a pair of trailing vortices is calculated.

Zbadano zagadnienie wytwarzania się warstwy wirów spływowych za płatem o dużym wydłużeniu. Uzyskane wyniki pozwalają wnioskować o pierwotnej strukturze wirów spływowych. Rozpatrzono problem rozpadu wirów i wyjaśniono istnienie prędkości osiowych w kierunku skrzydła. Obliczono wpływ struktury wewnętrznej na prędkości wzrostu wzajemnej niestęczności pary wirów spływowych.

Исследована проблема образования слоя вихрей течения за крылом с большим удлинением. Полученные результаты позволяют сделать вывод о первичной структуре вихрей течения. Рассмотрена проблема распада вихрей и выяснено существование осевых скоростей в направлении крыла. Вычислено влияние внутренней структуры на скорости роста взаимной неустойчивости пары вихрей течения.

1. Introduction

TRAILING vortices are formed by the roll up of the vorticity shed from the trailing edge of a lifting wing or hydrofoil. PRANDTL, BETZ and others did much valuable work in the 1930's, but the problem ceased to be one of the leading areas of aerodynamic research. Interest has recently been stimulated by the advent of jumbo jets, and the realisation that the vortices left by these aircraft may be a serious hazard, and active research is now in progress at many institutions.

Progress has been made by introducing simplifying assumptions which enable different parts of the flow to be discussed separately. The first assumption is that the lift distribution on the wing and the initial strength of the trailing vortex sheet is independent of the roll up process. Then for light loading, the lift on the wing can be studied by lifting surface theory. Unfortunately, lifting surface theory is not complete. There is a substantial body of numerical work, but analytical results are scarce, particularly for the loading near the tips which (as will be seen below) controls the initial rate of roll up and the core structure of the vortices.

Some analytical work has been done for the circular wing, but the practical case of the rectangular wing is a mystery. Because of boundary layer separation at the wing tips, it may not make physical sense to use lifting surface theory in this region. Flaps, engine struts, sweepback, etc. add major complications. All such questions will be bypassed here, and we shall examine the roll up on the assumption that the wing loading is known and,

moreover, is approximately elliptic, i.e.

$$(1.1) \quad \Gamma(x) = \frac{2\Gamma_0}{b}(bx-x^2)^{\frac{1}{2}}, \quad 0 < x < b.$$

Here, $\Gamma(x)$ is the strength of the bound vortex at spanwise coordinate x , Γ_0 is the root circulation, and b is the wing span. For a thin wing of root chord c and large aspect ratio moving with velocity U at angle of attack α , Prandtl's lifting line theory gives

$$(1.2) \quad \Gamma_0 = \pi U c \alpha.$$

It is a matter of further study to determine how well a particular wing is approximated by (1.1) and (1.2). Effects of finite aspect ratio can be included by replacing α by an effective angle of attack $\alpha - \Gamma_0/(2bU)$, which is exact for elliptic loading. The methods for studying roll up can be applied to fairly general load distributions, and there is not a serious loss of generality in restricting attention to the case of elliptic loading.

The lift on the wing is $L = \frac{1}{4}\pi Ub\Gamma_0$, and the induced drag is $D_i = \frac{1}{8}\pi\Gamma_0^2$. (These results are of course independent of (1.2). The density is given the value of unity.)

The coordinate system is chosen with x spanwise, y vertically downwards (to eliminate some inconvenient minus signs) and z parallel to the free stream. The velocity components are $(u, v, w + U)$.

The initial strength $\kappa(x)$ of the vortex sheet extending downstream of the trailing edge of the wing is $\kappa(x) = d\Gamma/dx$. The edges of the sheet roll up into two spirals under the action of the self induced velocities. Prandtl's classic sketch has been reproduced many times. The calculation of the roll up is an intractable problem of steady three-dimensional flow. The approximation is therefore made of replacing the steady three-dimensional flow by an equivalent unsteady two-dimensional flow in the xy plane, with $t = z/U$, because the two-dimensional flow is easier to study. The approximation neglects bending of the vortex lines and induced velocities parallel to the free stream. However, it is intuitively plausible and can be justified formally for light loading (i.e. $\alpha \rightarrow 0$) and large z ; see MOORE & SAFFMAN, 1973, Appendix A. It is of uncertain validity close to the wing tips, but the uncertainty here is coupled with uncertainty in the wing loading. There have been attempts to calculate numerically the steady three-dimensional shape of the sheet, but it is difficult to believe in their accuracy, since the simpler unsteady two-dimensional problem cannot yet be solved numerically with any confidence.

2. The roll up

We consider the unsteady two-dimensional motion of a vortex sheet which at $t = 0$ lies along the x -axis from o to b , and has an initial strength $\kappa(x)$. The motion is supposed inviscid, effects of viscosity being deferred to § 3. The equation of the sheet at subsequent times is given by the solution of a non-linear singular integro-differential equation. Introduce a Lagrangian coordinate Γ by the relation

$$\Gamma = \int^s \kappa(s, t) ds, \quad \kappa = \frac{d\Gamma}{ds},$$

where $\kappa(s, t)$ is the strength of the sheet at time t as a function of arc length s . Γ is the circulation about length s of the sheet. Let $X(\Gamma, t)$, $Y(\Gamma, t)$ be the parametric representation of the sheet, and $Z = X + iY$. Then

$$(2.1) \quad \frac{\partial}{\partial t} Z^*(\Gamma, t) = -\frac{i}{2\pi} \int \frac{d\Gamma'}{Z(\Gamma, t) - Z(\Gamma', t)},$$

where the asterisk denotes complex conjugate and the bar through the integral sign means that the Cauchy principal value is to be taken. For elliptic loading, the initial condition is

$$(2.2) \quad Z = \frac{b}{2} \pm \frac{b}{2} \left(1 - \frac{\Gamma^2}{\Gamma_0^2}\right)^{\frac{1}{2}}, \quad 0 \leq \Gamma \leq \Gamma_0.$$

Work is in progress at the California Institute of Technology to solve (2.1) with the initial condition (2.2), but the calculation is hard and progress is slow. The usual method of calculating vortex sheet motion is to replace the sheet by discrete line vortices whose motion can be obtained by integrating a set of ordinary differential equations. However, it has been shown (MOORE, 1971) that this approach is unsound, and leads to a jumbled chaotic mess when the equations are solved accurately. The more vortices are employed, the more chaotic is the result. Smooth roll up can be obtained only by some artificial ad hoc modification of the calculation (KUWAHARA & TAKAMI, 1973; MOORE, 1974) with an unknown error.

Note that

$$(2.3) \quad Z(\Gamma, t) = \frac{b}{2} \pm \frac{b}{2} \left(1 - \frac{\Gamma^2}{\Gamma_0^2}\right)^{\frac{1}{2}} + i \frac{\Gamma_0}{b} t$$

is an exact solution of (2.1) except at $\Gamma = 0$, and describes the sheet moving steadily downwards with velocity Γ_0/b . However, the failure at $\Gamma = 0$ is unacceptable and invalidates the solution. As can be shown easily, the momentum flux through infinitesimal circles centered on the ends $x = 0$, b does not vanish, and hence the solution (2.3) requires the imposition of external forces. In fact, (2.3) makes the right-hand side of (2.1) infinite at $\Gamma = 0$, so that the ends of the sheet are subject at the initial instant to infinite accelerations and they actually roll up instantaneously into spirals of infinite length. The solution of (2.1) for $t > 0$ must satisfy the requirement that the velocity of the ends ($\Gamma = 0$) should be finite, and this can only be accomplished by having the strength of the sheet vanish there.

Information about the flow near the ends of the sheet and the structure of the central parts of the spiral comes from a study of a semi-infinite vortex sheet with initial strength $\gamma x^{-\frac{1}{2}}$, $0 \leq x \leq \infty$. This means solving (2.1) with initial condition $Z = \frac{1}{4} \Gamma^2 / \gamma^2$. The solution

of this problem, with $\gamma = \Gamma_0/b^{\frac{1}{2}}$, clearly describes the form of the sheet near $\Gamma = 0$ in the initial stages of roll up. This problem was studied by KADEN (1931), and later by STERN (1956) and MANGLER & WEBER (1967). Their results can be expressed in the present notation by

$$(2.4) \quad Z(\Gamma, t) \sim Z_T(t) + \frac{\Gamma^2}{4\gamma^2\lambda} \exp\left(\frac{8i\gamma^4\lambda^2 t}{\pi\Gamma^3}\right)$$

as $\Gamma \rightarrow 0$. Here Z_T is the position of the spiral tip, and λ is a constant to be determined. Referred to polar coordinates centered on Z_T , the sheet is the spiral

$$(2.5) \quad r \sim \left(\frac{\gamma t \lambda^2}{\pi} \right)^{\frac{2}{3}} \theta^{-\frac{2}{3}} \sim \frac{\Gamma^2}{4\gamma^2 \lambda}.$$

Note that the spiral is infinitely long and, moreover, the strength of the sheet vanishes like $\frac{2}{3}\pi r/t$ as $r \rightarrow 0$.

A simple physical derivation of the spiral shape can be given. The distance between neighboring turns decreases as $r \rightarrow 0$, so the spiral tightens towards its center. The vorticity distribution is approximately axisymmetric and the velocity field is tangential and of magnitude $\Gamma/2\pi r$. Thus a fluid particle, at value Γ on the sheet, moves so that

$$(2.6) \quad \frac{dr}{dt} \doteq 0, \quad \frac{d\theta}{dt} \doteq \frac{\Gamma}{2\pi r^2}.$$

Let this fluid particle be initially at distance $x = \frac{1}{4}\Gamma^2/\gamma^2$ from the tip. By dimensional analysis, $r = x \text{fn}(\Gamma^3/t\gamma^4) = x/\lambda$, where λ is a constant by virtue of (2.6). The quantity λ can be thought of as a contraction factor, its physical significance being that the vorticity originally within a distance x of the tip is after roll up in a circle of radius x/λ about the tip. Thus λ measures the extent to which the vorticity is concentrated. Then (2.6) integrates to $r \doteq \frac{1}{4}\Gamma^2/\lambda\gamma^2$, $\theta \doteq \frac{1}{2}\Gamma t/\pi r^2$, which reduces immediately to (2.5).

(Further terms in the asymptotic expansion were obtained by STERN and by MANGLER & WEBER. The next term gives

$$r \sim \left(\frac{\gamma t \lambda^2}{\pi} \right)^{\frac{2}{3}} \theta^{-\frac{2}{3}} \left[1 + \frac{53\pi^2}{243\theta^2} + \dots \right].$$

However, the constant λ and the position Z_T are not determined by the analysis. This result can also be obtained from the integral equation (2.1). The first term is trivial, but the second term is a hard calculation. However, we wish to point out that the study of the integral equation (done in collaboration with Dr. D. W. MOORE) suggests the existence of terms of a different character in the complete asymptotic expansion corresponding to elliptical deformation of the spiral and the result is suspect. The analysis is, however, still incomplete. As regards Z_T , it follows from dimensional analysis that $Z_T \propto (\gamma t)^{\frac{2}{3}}$. KADEN made an estimate of the coefficient of proportionality by appealing to the conservation of impulse, which in the present context leads to

$$\int_0^\infty \left(Z - \frac{\Gamma^2}{4\gamma^2} \right) d\Gamma = -\frac{1}{4}\pi t \gamma^2 t,$$

and in effect substituting a rough approximation for Z .)

For $r \gg (\gamma t)^{\frac{2}{3}}$, the sheet is approximately straight and undisturbed, with strength $\gamma r^{-\frac{1}{2}}$. Thus there is a value of r ($\propto (\gamma t)^{\frac{2}{3}}$) for which the strength of the sheet is a maximum, the

maximum value being proportional to $(\gamma^2/t)^{\frac{1}{3}}$. The constants of proportionality are not yet known. It is to be kept in mind that the roll up process may be unstable, the mechanism being the Kelvin-Helmholtz instability of vortex sheets. (See PIERCE, 1961, for impressive pictures of this instability in the vortex sheets formed by impulsive motion over sharp edges.) However, there are reasons for believing that the instability may be suppressed by the increasing length of the sheet. For a straight sheet of strength κ , the amplitude A of a disturbance of wavelength $2\pi/n$ grows according to the equation

$$\frac{1}{A} \frac{dA}{dt} = \frac{\kappa n}{2}.$$

We estimate the growth rate of short waves on the spiral by using this equation with appropriate values of κ and n . Waves move with the mean of the velocity on the two sides of the sheet, i.e. crests have constant Γ . Then κ is proportional to r/t and $2\pi/n \propto ds/d\Gamma = \kappa^{-1}$. Thus, putting in the details,

$$(2.7) \quad \frac{1}{A} \frac{dA}{dt} = \frac{2\pi^2 r^2}{9t^2} \left(\frac{n}{\kappa} \right)_0,$$

where suffix 0 denotes original value. The growth rate is reduced and An , the wave slope tends to zero as $t \rightarrow \infty$. The disturbances, which can be expected to arise first at the position of maximum κ , are damped out as the spiral continues to tighten.

The constant λ is physically important as it determines the strength of the vorticity in the rolled up spiral. There is no completely satisfactory way of calculating λ ; the best that can be done is to follow KADEN (1931) and use an approximation due to BETZ (1932). This assumes that the radius of gyration of the vorticity inside a circle of radius r centered on Z_T is equal to the radius of gyration of the same vorticity when in its initial position on a straight line of length λr . The vorticity external to that being considered is being assumed to have negligible effect on the angular momentum of the vorticity ending up inside r . A simple calculation leads to

$$(2.8) \quad \lambda = 1.5.$$

No error estimate is known, but as there is nothing better, we shall use this value henceforth.

We now turn to the question of estimating the roll up of a finite sheet. The Betz approximation can be used to give an approximation for the final state of two completely rolled up, axisymmetric, trailing vortices, for arbitrary wing load distributions. But we want to know the rate at which the roll up takes place and the vortex strength in the intermediate stages of roll up. For this purpose, we approximate the elliptic loading by a distribution

$$(2.9) \quad \begin{aligned} \Gamma &= 2\gamma x^{\frac{1}{2}}, & 0 < x < \frac{1}{4}\Gamma_0^2/\gamma^2, \\ &= \Gamma_0, & \frac{1}{4}\Gamma_0^2/\gamma^2 < x < b - \frac{1}{4}\Gamma_0^2/\gamma^2, \\ &= 2\gamma(b-x)^{\frac{1}{2}}, & b - \frac{1}{4}\Gamma_0^2/\gamma^2 < x < b. \end{aligned}$$

(Note that if the wing were not elliptically loaded, we can still introduce the approximation (2.9), with if desired a different power law dependence, with the constants to be found from experiment or otherwise. But for simplicity, we stick to elliptic loading, so that $\gamma = \Gamma_0/b^2$, and (1.2) gives Γ_0 . MOORE & SAFFMAN (1973) suggested that $\gamma = 2\Gamma_0/\pi c^2$ might be appropriate for rectangular wings, but the evidence for this value is less strong now than was once thought.)

We now assume that beyond the first vertical tangent, i.e. $\theta > \pi$ for the left-hand vortex, the sheet has the shape given by the asymptotic formula (2.4) and (2.5). Moreover, we shall say that this part of the sheet constitutes the partially rolled up vortex. Then at time t , the radius r_V and strength Γ_V of the rolled up vorticity are given by putting $\theta = \pi$ in (2.5), i.e.

$$(2.10) \quad r_V = \frac{\gamma^{\frac{2}{3}} \lambda^{\frac{1}{3}} t^{\frac{2}{3}}}{\pi^{4/3}}, \quad \Gamma_V = 2\gamma \lambda^{\frac{1}{2}} r_V^{\frac{1}{2}}.$$

These expressions hold until

$$(2.11) \quad t = \frac{\pi^2}{4\lambda^2} \frac{\Gamma_0^3}{2\gamma^4} = \frac{\pi^2}{4\lambda^2} t^*, \quad \text{say,}$$

$$\Gamma_V = \Gamma_0,$$

$$r_V = \frac{1}{4} \Gamma_0^2 / \gamma^2 \lambda = R, \quad \text{say.}$$

After this time, the vorticity is said to be fully rolled up in trailing vortices of radius R and strength Γ_0 . We can write the state of the partially rolled up vortex as

$$(2.12) \quad \frac{\Gamma_V}{\Gamma_0} = \left(\frac{2\lambda}{\pi}\right)^{\frac{2}{3}} \left(\frac{t}{t^*}\right)^{\frac{1}{3}}, \quad \frac{r_V}{R} = \left(\frac{2\lambda}{\pi}\right)^{\frac{4}{3}} \left(\frac{t}{t^*}\right)^{\frac{2}{3}}.$$

Since $2\lambda/\pi$ is close to one, the time scale $t^* = \Gamma_0^3/2\gamma^4$ gives the characteristic time for roll up.

The completely rolled up vortices descend with a speed $\Gamma_0/2\pi b'$, where their horizontal separation b' is determined by conservation of linear momentum, i.e.

$$\Gamma_0 b' = \int_0^b \Gamma(x) dx,$$

where $\Gamma(x)$ is here the initial loading. For elliptic loading, $b' = \frac{1}{4}\pi b$, and t^* is the time for the vortices to descend a distance b/π^2 .

In terms of distance behind a wing for roll up to be complete, $Ut^* = \frac{1}{2}b^2/\pi c\alpha$, and depending on the angle of attack and aspect ratio of the wing, this distance may be 25–100 chord length. On the other hand, 50% roll up takes 1/8th of the distance or time, i.e. 3–12 chord lengths, the remainder of the vorticity still being in the unrolled up sheet.

MOORE (1974) estimated the rate of roll up of an elliptically loaded vortex sheet using a discrete vortex representation with an ad hoc amalgamation at the spiral center to avoid chaotic behavior and produce a smooth roll up. His results are fitted quite well by the estimate (2.12), particularly for small t . The numerical results do not have, of course, the kink at $t = t^*$.

Our simple model neglects the deformation of one vortex by the other, and elliptical self-deformation is also ignored. It is believed that effects are not important. The deformation of a vortex by a uniform strain was studied by MOORE & SAFFMAN (1971), and it was found that line vortices are remarkably resistant to planar deformations.

A measure of the error is provided by the failure to conserve energy. Since viscosity is neglected, and the motion is assumed to be two-dimensional, the kinetic energy should be conserved exactly and be equal to D_i , the induced drag. For two circular vortices of radius R , strength $\pm\Gamma_0$, centers b' apart, and axisymmetric circulation distribution $\Gamma(r)$, the kinetic energy is

$$(2.13) \quad \frac{\Gamma_0^2}{2\pi} \log \frac{b'}{R} + \frac{1}{2\pi} \int_0^R \frac{\Gamma^2}{r} dr = D_i.$$

We evaluate (2.13) with $\Gamma(r) = 2\gamma\lambda^{\frac{1}{2}}r^{\frac{1}{2}}$ (i.e. the discrete structure of the vortex is neglected) and obtain

$$D_i = \frac{\Gamma_0^2}{2\pi} (1 + \log \pi \lambda) = 1.03 \left(\frac{1}{8} \pi \Gamma_0^2 \right)$$

if $\lambda = 1.5$. This indicates an acceptable error of about 3%. (Alternatively, we could have used $\lambda = 1.4$ to ensure energy conservation.)

PRANDTL (see DURAND, 1934) used (2.13) in the opposite way to calculate R , given solid body rotation in the vortices. The result is $R = 0.28b$ for elliptic loading, somewhat larger than our estimate (2.11) which gives $R = 0.17b$. The velocity distributions in the two models are completely different. In particular, our model gives a tangential velocity

$$(2.14) \quad v_\theta = \frac{\Gamma}{2\pi r} = \frac{\gamma\lambda^{\frac{1}{2}}}{\pi r^{\frac{1}{2}}}, \quad r < r_v \leq R,$$

without a finite maximum (the singularity will be removed by viscosity). It can be argued that viscosity and/or turbulence would tend to produce solid body rotation, but in this case it is not permissible to assume that energy is conserved.

The main doubt about the application of our simple model to trailing vortices is the neglect of three-dimensional effects. It is known that the vortices contain significant axial velocities towards the wing, but we do not yet know how to calculate their effect on the roll up. On the other hand, given the roll up, we can estimate the axial velocities by using Bernoulli's equation.

With viscosity neglected,

$$(2.15) \quad p + \frac{1}{2}u^2 + \frac{1}{2}v^2 + \frac{1}{2}(U+w)^2 = H = p_\infty + \frac{1}{2}U^2$$

in the steady three-dimensional flow past the wing. We can now calculate w by assuming that u , v and p are given by the unsteady two-dimensional roll up problem, with $t = z/U$. Unfortunately, the simple model is not good enough to determine p with accuracy, except near the centers of the vortices, and numerical work (as yet undone) is needed for a good estimate. As $r \rightarrow 0$,

$$u^2 + v^2 \sim \frac{\gamma^2 \lambda}{\pi^2 r}, \quad \frac{\partial p}{\partial r} \sim \frac{\gamma^2 \lambda}{\pi^2 r^2}, \quad p \sim -\frac{\gamma^2 \lambda}{\pi^2 r}$$

and hence

$$(2.16) \quad w \sim \frac{\gamma^2 \lambda}{2\pi^2 U r}.$$

Thus infinite axial velocities *away* from the wing are predicted inside the trailing vortices. (The prediction that inviscid roll up would produce axial velocities away from the wing was made first by BATCHELOR, 1964. His velocities were finite because his vortices were assumed to be in solid body rotation.) In the next section, we shall give a rough estimate of the axial velocity distribution throughout the vortex, and more remarkably show that for elliptic loading, viscosity reverses the axial flow towards the wing while removing the singularity at $r = 0$.

3. Structure and decay of laminar trailing vortices

The processes of formation and decay are not completely independent, since decay starts before roll up is complete. But the decay is confined to a viscous core, and the interaction is weak and can be neglected. Thus to study the decay of one of the trailing vortices, we assume that the inviscid roll up is completed instantaneously, and that at station $z = 0$ we have an axisymmetric vortex with structure

$$(3.1) \quad \begin{aligned} v_\theta &= \frac{\gamma \lambda^{\frac{1}{2}}}{\pi r^{\frac{1}{2}}}, & v_r &= 0, & \text{for } r \leq R, \\ v_\theta &= \frac{\Gamma_0}{2\pi r}, & v_r &= 0, & \text{for } r > R. \end{aligned}$$

Cylindrical polar coordinates (r, θ, z) are employed, with velocity components $(v_r, v_\theta, U+w)$. The interaction of the two vortices is ignored. In addition, the discrete structure of the rolled up vortex is neglected, because the distance between the turns decreases as t increases and in fact viscosity merges the turns of the spiral quite rapidly and makes the approximation physically realistic. For elliptic loading, $R = \frac{1}{6} b$.

The equations of motion which describe the down-stream decay are

$$(U+w) \frac{\partial v_r}{\partial z} + v_r \frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r} + \nu \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2} \right),$$

$$(U+w) \frac{\partial v_\theta}{\partial z} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} = \nu \left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} + \frac{\partial^2 v_\theta}{\partial z^2} \right),$$

$$(U+w) \frac{\partial w}{\partial z} + v_r \frac{\partial w}{\partial r} = -\frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right),$$

$$\frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{\partial w}{\partial r} = 0.$$

The initial conditions (3.1) are not sufficient and further approximations are made. It is recognized that decay is slow and $\partial/\partial z \ll \partial/\partial r$, so the viscous terms with $\partial^2/\partial z^2$ are dropped. The equations are now parabolic and only the initial value of w is required to specify the problem. This is found by assuming that the pressure is initially in balance with the centrifugal force, i.e.

$$p = p_\infty - \frac{\Gamma_0^2}{8\pi^2 r^2}, \quad r > R,$$

$$p = p_\infty - \frac{\Gamma_0^2}{8\pi^2} \left(\frac{2}{Rr} - \frac{1}{R^2} \right), \quad r < R.$$

The initial value of w now follows from Bernoulli's equation (2.15),

$$w = 0, \quad r > R,$$

$$(3.2) \quad w = \left\{ U^2 + \frac{\gamma^2 \lambda}{\pi^2} \left(\frac{1}{r} - \frac{1}{R} \right) \right\}^{\frac{1}{2}} - U; \quad r < R.$$

Because the pressure in the vortex center is initially negatively infinite, there is a case for studying compressibility effects, but this will not be done here. A more important matter is the loss of total head in the fluid that has passed through the wing boundary layers. It is commonly believed that velocities towards the wing are caused by the reduction of w , due to boundary layer retardation, dominating the increase due to inviscid acceleration as given by (3.2). This is not so, and we shall therefore neglect boundary layer retardation for the present in order to emphasize that the decay process can lead by itself to $w < 0$ close to the wing. (It is known (BATCHELOR, 1964) that viscous decay makes $w < 0$ in the far wake, but this is the region for $(\nu z/U)^{\frac{1}{2}} \gg R$, i.e. $z/c \gg \frac{1}{36} (b^2/c^2) \text{Re}$, where $\text{Re} = Uc/\nu$ is the Reynolds number based on chord. The far wake is outside the range of experiment, and observations of negative w in distances of order Ut^* are not explained by Batchelor's theory of the far wake.)

The non-linear problem posed here is currently under study by numerical methods, but results are not yet available. MOORE & SAFFMAN (1973) examined a quasi-linear approximation valid for light loading. This is

$$(3.3) \quad -\frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r},$$

$$(3.4) \quad U \frac{\partial v_\theta^2}{\partial z} = \nu \left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} \right),$$

$$(3.5) \quad U \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right).$$

It is now convenient to write $t = z/U$, and describe the decay in terms of time rather than downstream distance.

For $(\nu t)^{\frac{1}{2}} \ll R$, which is the case for distances up to many thousands of chord lengths, depending on the aspect ratio and Reynolds number, the initial conditions for (3.3)–(3.5) can be written

$$(3.6) \quad v_\theta = \frac{\beta}{r^{\frac{1}{2}}}, \quad w = \frac{\beta^2}{Ur}, \quad 0 < r < \infty,$$

where $\beta = \gamma \lambda^{\frac{1}{2}}/\pi$. The solution for v_θ is

$$(3.7) \quad v_\theta = \frac{\beta}{2^{\frac{3}{2}}} \Gamma\left(\frac{5}{4}\right) \frac{r}{(\nu t)^{\frac{3}{4}}} M\left(\frac{3}{4}; 2; -\frac{r^2}{4\nu t}\right),$$

where M is the confluent hypergeometric function, also denoted by ${}_1F_1$. The maximum value of v occurs at

$$(3.8) \quad r_1 = 2.92(\nu t)^{\frac{1}{2}} = 2.92 c(z/c)^{\frac{1}{2}} \text{Re}^{-\frac{1}{2}},$$

and the maximum value of v is $0.49\beta(\nu t)^{-\frac{1}{4}}$.

The distance r_1 is a natural definition of the radius of a trailing vortex, and experimenters have usually taken the position of maximum tangential velocity as defining the vortex radius. Considerations of inviscid roll up led in § 2 to a radius r_ν , see equation (2.12), which is substantially larger. For elliptic loading,

$$\frac{r_1}{r_\nu} = 2.92 \left(\frac{\pi}{\alpha}\right)^{\frac{2}{3}} \left(\frac{b}{c}\right)^{\frac{1}{3}} \text{Re}^{-\frac{1}{2}} \left(\frac{c}{z}\right)^{\frac{1}{6}}.$$

This explains why observed radii are much smaller than those predicted by Prandtl's theory. The part of the vortex for $r < r_1$ will be called the core.

The axial velocity can be expressed as integrals of confluent hypergeometric functions. We write

$$(3.9) \quad w = \frac{\beta^2}{U(\nu t)^{\frac{1}{2}}} W\left(\frac{r^2}{4\nu t}\right).$$

Then W is a dimensionless function of its argument, and has been calculated, see MOORE & SAFFMAN (1973). We find that $W < 0$ for $r < 1.4(\nu t)^{\frac{1}{2}}$, $W(0) = -0.13$. Thus not only does viscosity remove the singularities at the center of the vortex predicted by the inviscid theory, but in doing so the direction of the axial velocity is reversed in sign. Outside the core, the inviscid mechanism gives $w \sim \beta^2/Ur > 0$.

(MOORE & SAFFMAN (1973) considered a more general wing loading for which the initial condition on v_θ gives $v_\theta = \beta r^{-n}$. This corresponds to a wing loading $\propto x^{1-n}$ near the tip. It was found that the value of w on $r = 0$ depends on n , being negative if $n > 0.44$ and positive if $n < 0.44$. A delta wing, low aspect ratio planform corresponds to the singular case $n = 0$, so the analysis is consistent with the known

results that $w > 0$ in the core of leading edge vortices produced by delta wings. Batchelor's analysis of the far wake is the other singular case $n = 1$, $\beta = \Gamma_0$, where the necessary extra condition of finite axial flux has to be imposed. As w at $r = 0$ was known to have different signs for the two limiting cases $n = 0$, $n = 1$, it is not surprising that reversal takes place at an intermediate value. The pressure in the core is proportional to $-\beta^2(U/\nu z)^n$ and the retardation is caused by the adverse pressure gradient working against the inviscid value of w which is $\frac{1}{2}\beta^2/U(-1+1/n)r^{-2n}$. It is clear, therefore, that the adverse pressure gradient is going to dominate as n goes from 0 to 1.)

MOORE & SAFFMAN (1973) also considered the axial velocity defect due to boundary layer retardation. In the quasi-linear approximation, it is sufficient to add w_δ to the initial value of w in (3.6). If $\delta_2(x)$ is the combined momentum thickness of the boundary layers on the wing at spanwise coordinate x , the interpretation of λ as a contraction factor gives

$$w_\delta = -\frac{\lambda U}{2\pi r}\delta_2(\lambda r).$$

For a rectangular wing, $\delta_2 = 1.33 c\text{Re}^{-\frac{1}{2}}$, and the additional axial velocity to be added to (3.9) at station z is

$$(3.10) \quad w_\delta(z) = -\frac{\lambda U \delta_2}{4\pi(\nu t)^{\frac{1}{2}}}\Gamma\left(\frac{1}{2}\right)M\left(\frac{1}{2}; 1; -\frac{r^2}{4\nu t}\right).$$

Note that w_δ is independent of Re . (A case can be made for using the displacement thickness δ_1 instead of δ_2 . The question is still open.)

In the core, the two effects add to give a larger velocity towards the wing. Outside the core, the two effects are in opposite direction. The inviscid acceleration mechanism wins or loses according as $5\alpha^2\text{Re}^{\frac{1}{2}}(c/b)$ is greater or less than unity.

To sum up, the laminar trailing vortex is essentially inviscid with radius r_ν and tangential velocity proportional to $r^{-\frac{1}{2}}$, with a viscous core containing axial velocities towards the wing. The decrease in the kinetic energy of the transverse flow due to viscosity appears as an increase in pressure and an axial flux towards the wing. The momentum flux equation gives for the drag,

$$D = \iint \left(\Delta H + \frac{1}{2}(u^2 + v^2) - \frac{1}{2}w^2 \right) dx dy,$$

where the integral is over any plane $z = \text{constant}$ downstream of the wing, and ΔH is the loss of total head. If boundary layer retardation is neglected, $D = D_i$. By definition,

$$\Delta H + \frac{1}{2}(u^2 + v^2) = p_\infty - p - Uw,$$

and it follows from (3.5) that the integral is constant ($\frac{1}{2}w^2$ being negligible when (3.5) holds). As the crossflow velocities tend to zero and $p \rightarrow p_\infty$, the induced drag is carried by the axial momentum flux.

The loss of transverse kinetic energy has been calculated. It is

$$(3.11) \quad \pi \int_0^{\infty} (\beta^2 - r v_0^2) dr = 8.6 \beta^2 (\nu t)^{\frac{1}{2}}.$$

4. The turbulent vortex

The trailing vortices may be turbulent, instead of laminar, with a different structure and decay rate. However, the main evidence for turbulent vortices is indirect, being the body of experimental data showing that the core radius r_1 grows significantly faster than the laminar value (3.8). If the observed growth rates are used to infer eddy viscosities ν_T , the values of ν_T/ν range from 10 to 1000 depending on the Reynolds number Γ_0/ν .

The experimental data are far from satisfactory, but suggest $\nu_T/\nu \propto (\Gamma_0/\nu)^{\frac{1}{4}}$, see OWEN (1970). Care must be taken to distinguish "genuine turbulence" from "apparent turbulence" caused by the random wandering of the laminar vortex due to wing flutter or free stream turbulence. Experiments recently carried out in a water tunnel at the California Institute of Technology show significant amounts of unsteadiness in the position of apparently laminar vortices.

There is no satisfactory treatment of the turbulent vortex. Here we summarize a recent attempt (SAFFMAN, 1973). Arguments were given for taking, as a model of the turbulent vortex, the mean circulation distribution

$$(4.1) \quad \Gamma = \Gamma_1 \frac{r^2}{r_1^2}, \quad r < r_1,$$

$$(4.2) \quad \Gamma = \Gamma_1 \left(\log \frac{r}{r_1} + 1 \right), \quad r_1 < r \ll r_0.$$

Further,

$$(4.3) \quad r_1 = k (\nu \Gamma_1)^{\frac{1}{4}} t^{\frac{1}{2}},$$

where k is a constant, roughly equal to two. The radius r_0 is the outer edge of the vortex, where $\Gamma = \Gamma_0$, and is related to Γ_1 by

$$(4.4) \quad r_0 = k' (\Gamma_1 t)^{\frac{1}{2}},$$

where k' is another constant of order $(2\pi)^{-\frac{1}{2}}$. The ratio Γ_1/Γ_0 is weakly dependent on Γ_0/ν ; it was estimated that

$$(4.5) \quad \frac{\Gamma_0}{\Gamma_1} \doteq \log \frac{r_0}{r_1}.$$

The tentative nature of these results should be kept in mind.

The defect region $r_1 < r < r_0$ is examined by postulating a profile and appealing to the conservation of angular momentum defect. Define

$$(4.6) \quad J(t) = \frac{1}{r_1^2} \int_0^{\infty} \left(1 - \frac{\Gamma}{\Gamma_0} \right) r dr.$$

It is an *exact* consequence of the equations of motion, for both laminar and turbulent flow, that

$$(4.7) \quad J(t) = \frac{A_0}{r_1^2} + \frac{2\nu t}{r_1^2},$$

where A_0 is a constant fixed by initial conditions.

For the laminar vortex, the initial condition (3.1) gives

$$(4.8) \quad A_0 = \frac{1}{10} R^2.$$

We assume as the profile in the defect region

$$(4.9) \quad \Gamma = \Gamma_0 + \Gamma_1 \left[\log \frac{r}{r_0} + 1 - \chi(t) \frac{r}{r_0} + \{\chi(t) - 1\} \frac{r^2}{r_0^2} \right].$$

(There was an oversight in SAFFMAN (1973). Equation (4.9) replaces equation (71) of that paper. The results reported earlier are unaffected.) With this profile, it follows that

$$(4.10) \quad J(t) = \frac{1}{12} \frac{\Gamma_1 r_0^2}{\Gamma_0 r_1^2} \chi,$$

terms of relative order r_1^2/r_0^2 being neglected. Hence, from (4.7) with molecular viscosity neglected,

$$(4.11) \quad \chi(t) = \frac{6}{5} \frac{R^2 \Gamma_0}{r_0^2 \Gamma_1} \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.$$

The distribution of circulation given by (4.9) develops an overshoot of circulation as t increases. It occurs when $\chi = 1$, i.e.

$$(4.12) \quad r_0 = \left(\frac{6\Gamma_0}{5\Gamma_1} \right)^{\frac{1}{2}} R, \quad t = \frac{6\Gamma_0 R^2}{5k'^2 \Gamma_1^2}.$$

The overshoot of circulation is a general feature of turbulent vortices and not dependent upon the model (GOVINDARAJU & SAFFMAN, 1971). For smaller values of r_0 , the circulation increases monotonically; for larger values, the point of maximum circulation asymptotes $r_0/2^{\frac{1}{2}}$.

The application of these results to the turbulent trailing vortex is not permissible until $r_0 \geq R$, as before this, the trailing vorticity is not completely rolled up, i.e.

$$(4.13) \quad t \geq \frac{\Gamma_0}{k'^2 \Gamma_1} \frac{R^2}{\Gamma_0} = \frac{1}{18} \frac{\Gamma_0}{k' \Gamma_1} t^*,$$

where t^* (§ 2) is the time for laminar roll up to be complete. Typical values of Γ_0/Γ_1 are around two. The value of k' is uncertain, but taking it to be roughly $1/2$, the distance to roll up of a turbulent vortex is about one-quarter of that for the laminar vortex. According to (4.12), the overshoot of circulation will develop at about twice the distance to turbulent roll up, and this will be before $r_0 \doteq \frac{1}{2} b$ and the two vortices interact. It is noteworthy that an overshoot of circulation implies that the mean velocity distribution

is unstable to axisymmetric disturbances leading to Taylor vortices with an axial wavelength of order r_0 . Besides axial variations of vortex structure, enhanced mixing between the core and the outer parts of the vortex will occur. If the core is marked with smoke, it may appear as if the vortex has "burst". There is some experimental evidence that this occurs at distances of the predicted magnitude (CHEVALLIER, 1973). Many casual observations of condensation trails show axial variations.

When the radius of the vortices is equal to the semi-span, it is appropriate to treat the two vortices together as constituting a two-dimensional turbulent cylinder of fluid. We shall not discuss this problem here.

The axial flow in a turbulent trailing vortex was also considered by SAFFMAN (1973). It was assumed that transfer of axial momentum by Reynolds stresses is negligible for $r > r_1$, so that (with light loading)

$$(4.14) \quad w = -\frac{1}{U} \{p(r; r_0) - p(r; r)\}, \quad (r < r_0),$$

where $p(r; r_0)$ is the pressure at radius r when outer radius is r_0 . This pressure is given by the centrifugal force balance,

$$(4.15) \quad p(r; r_0) = -\frac{\Gamma_0^2}{8\pi^2 r_0^2} - \frac{1}{4\pi^2} \int_r^{r_0} \frac{\Gamma^2}{r^3} dr.$$

Substituting the profile (4.9), we obtain after some algebra

$$(4.16) \quad w(r_1) = -\frac{\Gamma_0^2}{8\pi^2 r_0^2} \left(1 - \frac{5\Gamma_1^2}{2\Gamma_0^2}\right),$$

plus smaller terms. This result is insensitive to the profile for $r \gg r_1$. It follows that the vortex contains velocities towards the wing if $\Gamma_1/\Gamma_0 < 0.632$. Arguments were given to show that $w = w(r_1)$ inside the core.

To this velocity should be added that due to retardation in the wing boundary layer. If this is distributed uniformly, there will be an extra velocity $w_s = -Uh\delta_2/r_0^2$, where δ_2 is the average momentum thickness at the trailing edge.

5. The mutual instability of trailing vortices

The trailing vortices of large aircraft constitute a serious hazard for following planes. It is therefore of considerable interest to determine if there are any processes which can accelerate the decay. Blowing from wing tips has been suggested, but its effectiveness is uncertain. If it made a laminar vortex turbulent, the decay would be enhanced, but otherwise there is no obvious reason why it should make any difference.

CROW (1970) discussed a further mechanism, namely the mutual instability of a pair of parallel line vortices. Crow employed the Biot-Savart law of induction with an ad hoc cut-off to obtain the equations of motion for the perturbed vortices. This procedure was justified rigorously by MOORE & SAFFMAN (1972) for the case that $|\bar{w}|R^2/\Gamma L \ll 1$, $R \ll L$, where $\pi\bar{w}R^2$ is the axial flux in the vortex (relative to the free stream) and L is the wavelength of the disturbance. MOORE & SAFFMAN gave equations for the spanwise (ξ_+ , ξ_-)

and vertical (η_+ , η_-) displacements of the vortices executing sinusoidal oscillations of wave length $2\pi/n$,

$$\frac{d\xi_+}{dt} = \frac{\Gamma}{2\pi B^2} \left[\eta_+ - (nBK_1 + n^2B^2K_0)\eta_- + \frac{1}{2}n^2B^2 \left(\frac{1}{2} - C - \log n - \log \delta \right) \eta_+ \right],$$

$$\frac{d\eta_+}{dt} = \frac{\Gamma}{2\pi B^2} \left[\xi_+ - nK_1B\xi_- - \frac{1}{2}n^2B^2 \left(\frac{1}{2} - C - \log n - \log \delta \right) \xi_+ \right],$$

with similar equations for ξ_- and η_- obtained by interchanging + and - and changing the sign of Γ . The quantity δ is related to the effective radius of the vortex, $\delta = \frac{1}{2}e^{\frac{1}{4}}R_{\text{eff}}$, where

$$(5.1) \quad R_{\text{eff}} = R \exp \left[\frac{1}{4} - \frac{2\pi^2 R^2}{\Gamma^2} \langle v^2 - 2w^2 \rangle \right],$$

angle brackets denoting average values through the interior of the vortex of radius R . The values of δ could be different for the two vortices if their structures were dissimilar. ($C = 0.5772 \dots$). The argument of the modified Bessel functions, K_0 and K_1 , in nB . B is the separation of the pair.

CROW supposed the vortices were in solid body rotation, as proposed by PRANDTL, so that for elliptic loading, $\Gamma = \Gamma_0$, $B = \pi b/4$, $\delta_+ = \delta_- = 0.126B$. Under the conditions of validity of the theory, symmetric disturbances ($\xi_+ = -\xi_-$, $\eta_+ = \eta_-$) are then unstable, and the most unstable wavelength is $7.4B$ and the e -folding time is $1.24(2\pi B^2/\Gamma_0) = 0.31\pi^3 t^*$. If we use the structure predicted by the theory of laminar decay equation (3.7), it is easily seen that the viscous core and axial velocity are negligible, leading to $R^2 \langle v_\theta^2 \rangle = \frac{1}{2}\Gamma_0^2/\pi^2$, $R_{\text{eff}} = e^{-\frac{3}{4}}R$, and $\delta_+ = \delta_- = 0.064B$. Now the most unstable wavelength is found to be $8.5B$ and the e -folding time is $1.21(2\pi B^2/\Gamma_0)$. The difference is negligible. Since the growth time scale is about $10t^*$, the fact that the trailing vortices are not completely rolled up before time t^* is unimportant. If, for the sake of illustration, it is supposed that the same analysis may be employed for $t < t^*$ with R and Γ replaced by r_V and Γ_V , the disturbance of wavelength $8.5B$ is neutrally stable only for $r_V/R < 1/25$.

The instability criterion does not depend on the internal structure of the vortex, but only on its radius R , for from equation (2.13),

$$(5.2) \quad R^2 \langle v_\theta^2 \rangle = \frac{D_i}{\pi} - \frac{\Gamma_0^2}{2\pi^2} \log \left(\frac{B}{R} \right).$$

The greater the induced drag, the smaller δ , and the more stable the vortices. As the induced drag increases as the speed decreases, the trailing vortices of landing aircraft will therefore be slightly more stable.

Unless the process of formation of turbulent vortices transfers significant amounts of energy from the free stream into the vortices⁽¹⁾, leading to an increase in $R^2 \langle v_\theta^2 \rangle$ over

(1) No criterion is known for determining if trailing vortices are laminar or turbulent. The energy of a turbulent vortex is more sensitive than the angular momentum to the assumed form of the defect law, and a better approximation than (4.9) is needed.

that given by (5.2), turbulent vortices should behave in the same way as laminar vortices with regard to the mutual instability. However, the time for turbulent vortices to interact through growth is comparable with t^* , and the mutual instability is probably masked by turbulent dissipation.

Observations of the mutual instability show that the vortices burst at the crests of the waves, where the separation is largest. This may be due to a transition from laminar to turbulent flow induced by the instability. A more plausible explanation is that the vortices are compressed at the crests, and that a line vortex is unstable to axisymmetric buckling when the compression rate is great enough. (The stability of a line vortex in axisymmetric stagnation point flow would be of interest; however, non-linear effects may be crucial.) As argued by MOORE & SAFFMAN (1972), internal waves tend to suppress changes in vortex radius due to unsteady displacements. The characteristic frequency of internal waves is $\Gamma_0/\pi R^2$, and buckling may therefore occur when

$$(5.3) \quad -\frac{\partial V_s}{\partial s} \approx \frac{\Gamma_0}{\pi R^2},$$

where V_s is the velocity along the vortex. MOORE (1972) has studied the finite amplitude oscillations of a pair of line vortices and shown that the linear theory is a good approximation for amplitudes large enough for the vortices to be touching. The maximum value of $-\partial V_s/\partial s$ occurs at the crests (see MOORE & SAFFMAN (1972), equation D3), so this is where buckling is most likely to occur.

CROW suggested that the mutual instability might be stimulated by periodic oscillations of lift and wing loading. This suggestion has been followed up by CHEVALLIER (1973). Some effect is observed, but analysis is rendered difficult by the spanwise vorticity in the wake due to the fluctuations in the wing loading.

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CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA.
