

## On nonlocal continuum theories of elasticity

D. ROGULA (WARSZAWA)

THIS PAPER is concerned with examining various possible approaches to nonlocal linear theory of elasticity of continuous media. As the governing equations of the theory an equation of the form

$$Lu = f$$

is assumed, where  $u$  and  $f$  are some tempered distributions and  $L$  is a linear operator. Apart from some general assumptions, no particular form of  $L$  is required. For homogeneous media, a classification of the operators  $L$  is given in terms of singular order which is either real number, or  $+\infty$  or  $-\infty$ . The fundamental solution is discussed and a theorem relating its singularity to the operator  $L$  is proved. The energy and interaction of point defects, modelled by force centers is analysed for various  $L$ .

Praca dotyczy badania rozmaitych form nielokalnej liniowej teorii ośrodków ciągłych. Przyjęto podstawowe równanie w postaci

$$Lu = f,$$

gdzie  $u$  i  $f$  są pewnymi dystrybucjami temperowanymi, a  $L$  jest operatorem liniowym. Operator ten spełnia pewne ogólne założenia, nie żąda się jednak jakiegось jego szczególnej postaci. Dla ośrodków jednorodnych podano klasyfikację możliwych operatorów  $L$  wprowadzając pojęcie rzędu osobliwości operatora; rząd osobliwości wyraża się bądź liczbą rzeczywistą, bądź też równa się  $+\infty$  lub  $-\infty$ . Przedyskutowano rozwiązanie podstawowe i udowodniono twierdzenie wiążące osobliwość tego rozwiązania z rzędem osobliwości operatora  $L$ . Dla różnych operatorów  $L$  przeanalizowano energię i oddziaływanie defektów punktowych, modelowanych przy pomocy odpowiednich centrów sił.

В работе исследуются различные формы нелокальной теории сплошных сред. Основное уравнение принято в виде

$$Lu = f$$

где  $u$  и  $f$  являются некоторыми обобщенными функциями медленного роста, а  $L$  является линейным оператором. Этот оператор удовлетворяет некоторым общим предположениям, не требуется, однако, какой-либо его специфической формы. Дана классификация допустимых операторов  $L$  для однородных сред, основанная на введении понятия порядка особенности оператора; порядок особенности является либо действительным числом, либо равен  $+\infty$  или  $-\infty$ . Обсуждено основное решение и доказана теорема, связывающая особенность этого решения с порядком особенности оператора  $L$ . Для различных операторов  $L$  произведен анализ энергии и взаимодействия точечных дефектов, моделируемых с помощью соответствующих центров сил.

### 1. Introduction — the integral theory

ONE MAY hope to achieve a pretty fair description of effects arising from nonlocality of atomic interactions in real bodies without giving up the idea of a continuous medium, if one chooses an appropriate integral equation as a governing equation of the theory. This is the way in which the nonlocal theory of elasticity is usually formulated (KRÖNER & DATTA, 1966, KRÖNER, 1967).

The governing equation can, in this case, be written as a linear integral relation between external forces  $f_i(\mathbf{x})$  and displacements  $u_i(\mathbf{x})$ :

$$(1.1) \quad \int \Phi_{ij}(\mathbf{x}, \mathbf{x}') u_j(\mathbf{x}') d^3x' = f_i(\mathbf{x}),$$

with an appropriate kernel  $\Phi_{ij}(\mathbf{x}, \mathbf{x}')$  which is a second order tensor function depending on two points  $\mathbf{x}$  and  $\mathbf{x}'$ . In the important case of a homogeneous medium, the kernel  $\Phi_{ij}$  depends on the difference  $\mathbf{x} - \mathbf{x}'$  only,

$$(1.2) \quad \Phi_{ij}(\mathbf{x}, \mathbf{x}') = \Phi_{ij}(\mathbf{x} - \mathbf{x}').$$

Alternatively, instead of (1.1) we can postulate an integral stress-strain relation

$$(1.3) \quad \sigma_{ij}(\mathbf{x}) = \int C_{ijkl}(\mathbf{x} - \mathbf{x}') u_{k,l}(\mathbf{x}') d^3x',$$

subsequently making use of equation

$$(1.4) \quad \sigma_{ij,j} + f_i = 0.$$

In this formulation, we have to choose the kernel  $C_{ijkl}(\mathbf{x} - \mathbf{x}')$  which is a fourth order tensor.

Roughly speaking, under additional conditions which must be incorporated into the theory, in order to make it sensible, these two formulations are equivalent. The disadvantage of the Eq. (1.3) is that it makes use of the concept of the stress tensor which, for long range interactions, has rather vague physical meaning, if any. The disadvantage of the Eq. (1.1), on the other hand, is that it makes no sense for dislocated bodies, when the displacement field is multi-valued. The last topic we shall discuss separately.

The use of the integral continuum theory may be justified by the following arguments:

a. let a typical interatomic distance be  $a$ , and a typical range of interatomic forces in a given material be  $l$ . The idea of a continuous medium may be expected to work at distances  $\lambda$  which are much greater than  $a$ ,

$$(1.5) \quad \lambda \gg a,$$

and validity of classical elasticity should be restricted to distances much greater than  $l$ ,

$$(1.6) \quad \lambda \gg l.$$

Thus, in the case

$$(1.7) \quad l \gg a,$$

there can exist an intermediate range of distances at which the idea of a continuous medium is applicable but classical elasticity is not.

b. The general form of the Eq. (1.1) is quite similar to that of the fundamental equation of the lattice theory. The latter can be written as

$$(1.8) \quad \sum_m \tilde{\Phi}_{ij}(\mathbf{x}_m, \mathbf{x}_{m'}) u_j(\mathbf{x}_{m'}) = \tilde{f}_i(\mathbf{x}_m),$$

where  $\mathbf{x}_m, \mathbf{x}'_m$  denote the equilibrium positions of the corresponding atoms,  $\tilde{\Phi}_{ij}(\mathbf{x}_m, \mathbf{x}'_m)$  represent the corresponding force constants, and  $\tilde{f}_i(\mathbf{x}_m)$  is an external force acting on the  $m$ -th atom.

This similarity leads one to expect a close relation between the force constants and the kernel of the Eq. (1.1). Even more, provided that the functions involved are sufficiently smooth on the atomic scale, one may expect to estimate the kernel  $\Phi_{ij}(\mathbf{x}, \mathbf{x}')$  in terms of force constants and, in this way, to derive the integral continuum theory from the lattice theory.

c. Provided that the displacement field  $u_j(\mathbf{x}')$  is sufficiently smooth it can be developed into Taylor's series:

$$(1.9) \quad u_j(\mathbf{x}') = u_j(\mathbf{x}) + \sum_{1 \leq |\mu| \leq N} \frac{1}{\mu!} \partial^\mu u_j(\mathbf{x}) \cdot (\mathbf{x}' - \mathbf{x})^\mu + \text{the remainder},$$

where

$$(1.10) \quad \mu! = \mu_1! \mu_2! \mu_3!$$

If for any reason the terms of orders higher than  $n$ , with  $n \ll N$ , are not important, they can be dropped. Substitution of the remaining terms into the Eq. (1.1) yields a differential equation of the form

$$(1.11) \quad a_{ij\mu} \partial^\mu u_j(\mathbf{x}) = f_i(\mathbf{x}),$$

the coefficients being equal to

$$(1.12) \quad a_{ii\mu} = \frac{1}{\mu!} \int \Phi_{ij}(\mathbf{x}, \mathbf{x}') (\mathbf{x}' - \mathbf{x})^\mu d^3x'.$$

This seems to justify the view that strain gradient theory can be looked upon as an approximation to the integral continuum theory.

However, as we shall see later, more thorough mathematical discussion does not completely support the above views. The range of applicability of the integral continuum theory is a more delicate matter and its relation to crystal lattice theory, on the one hand, and to strain gradient theory, on the other, will show themselves to be more complicated.

We shall begin from discussion of an example which has been given by BARNETT (1969). It will illustrate some difficulties which may be encountered in the integral continuum theory.

## 2. A troublesome integral equation

Consider an integral equation of the form (1.1) with the kernel

$$(2.1) \quad \Phi_{ij}(\mathbf{x}, \mathbf{x}') = c_{iljm} \partial_l \partial'_m \Phi(\mathbf{x} - \mathbf{x}'),$$

where  $C_{iljm}$  is the classical isotropic tensor,

$$(2.2) \quad c_{iljm} = \mu(\delta_{ij}\delta_{lm} + \delta_{im}\delta_{jl}) + \lambda\delta_{il}\delta_{jm},$$

and the function  $\Phi(\mathbf{x} - \mathbf{x}')$  is given by

$$(2.3) \quad \Phi = \left( \frac{\beta}{\sqrt{\pi}} \right)^3 e^{-\beta^2(\mathbf{x} - \mathbf{x}')^2}.$$

We can also put this equation in the form (1.3) with

$$(2.4) \quad C_{ijkl}(\mathbf{x}, \mathbf{x}') = c_{ijkl} \Phi(\mathbf{x} - \mathbf{x}').$$

For this form of  $\Phi$ , the parameter  $\beta^{-1}$  can be interpreted as the range of interactions. The numerical factor in (2.3) is chosen so that when  $\beta^{-1} \rightarrow 0$ , then

$$(2.5) \quad \Phi(\mathbf{x} - \mathbf{x}') \rightarrow \delta^{(3)}(\mathbf{x} - \mathbf{x}'),$$

and the corresponding equations become purely classical.

Now, let us try to find the fundamental solution  $G_{ij}(\mathbf{x})$ :

$$(2.6) \quad \Phi_{ij}(\mathbf{x} - \mathbf{x}') G(\mathbf{x}') = \delta^{(3)}(\mathbf{x}) \delta_{in}.$$

Making use of the Fourier transformation, we obtain the equation

$$(2.7) \quad \hat{\Phi}_{ij}(\mathbf{k}) \hat{G}_{jn}(\mathbf{k}) = \delta_{in}$$

for the corresponding Fourier transforms. The Fourier transform of  $\Phi_{ij}$  can easily be calculated,

$$(2.8) \quad \hat{\Phi}_{ij}(\mathbf{k}) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \Phi_{ij}(\mathbf{x}) = [\mu k^2 \delta_{ij} + (\lambda + \mu) k_i k_j] e^{-k^2/4\beta^2},$$

and from the Eq. (2.7), we obtain:

$$(2.9) \quad \hat{G}_{ij}(\mathbf{k}) = \left[ \frac{1}{\mu} \left( \frac{\delta_{ij}}{k^2} - \frac{k_i k_j}{k^4} \right) + \frac{1}{\lambda + 2\mu} \frac{k_i k_j}{k^4} \right] e^{k^2/4\beta^2}.$$

For  $1/\beta = 0$ , this is the classical result. Otherwise, however,  $\hat{G}_{ij}(\mathbf{k})$  has an exponential growth at infinity and cannot be retransformed in a usual way. Therefore, we have to conclude that, in the example considered, a Fourier-transformable fundamental solution does not exist.

From the mathematical point of view, the meaning of the last statement is not quite clear. It can be made precise in terms of tempered distributions. The tempered distributions are defined (see e.g. HÖRMANDER, 1964) as continuous linear forms on the space  $S$  which consists of infinitely differentiable functions  $\varphi(\mathbf{x})$  such that

$$(2.10) \quad \sup_{\mathbf{x}} |x^\mu \partial^\nu \varphi| < \infty$$

for any two multi-indices  $\mu, \nu$ . It can easily be proved that, if  $\varphi \in S$  and  $u$  is a tempered distribution, then the convolution of  $\varphi$  and  $u$  exists, and is again a tempered distribution. Moreover, the Fourier transform of this convolution equals the product of the Fourier transforms  $\hat{\varphi}$  and  $\hat{u}$ , and is a tempered distribution, too. Thus, by observing that the kernel defined by the Eqs. (2.1)–(2.3) belongs to the space  $S$ , we see that the Eq. (2.1) becomes meaningful in the sense of convolution for any  $u_i$  which is a tempered distribution. In this case, the Fourier transformation method we have just applied to solve this equation is entirely justified. The result (2.9), being itself no tempered distribution, cannot be retransformed into a tempered distribution. Therefore, the rigorous conclusion is that there is no tempered distribution which, in the example considered, could serve as a fundamental solution.

Intuitively, the class of tempered distributions consists of those distributions which do not grow too fast at infinity. Thus, even if we were able to find a solution in the class of all distributions, it would not be physically acceptable because of its behaviour at infinity.

Non-existence of a good fundamental solution, being a disadvantage from the point of view of calculational efficiency of the theory, might however be thought to be due to too singular a character of the  $\delta$ -type forces. For smooth forces, the theory might still be expected to work and yield smooth solutions.

To see that this is not exactly the case, let us consider an example of forces

$$(2.11) \quad f_i = -\partial_i \psi,$$

where

$$(2.12) \quad \Psi = \left( \frac{\alpha}{\sqrt{\pi}} \right)^3 e^{-\alpha^2 r^2}$$

is a function similar to  $\Phi$  but with a different parameter  $\alpha$ . These forces are central, with no resultant force or moment, and their magnitude as a function of the distance  $r$  is

$$(2.13) \quad f = 2\alpha^2 r e^{-\alpha^2 r^2} \left( \frac{\alpha}{\sqrt{\pi}} \right)^3.$$

The force field (2.11) is infinitely differentiable and, if  $\alpha^{-1} \gg a$ , describes perfectly smooth distribution of forces on the atomic scale.

Now, applying the Fourier transformation, instead of the Eq. (2.7) we obtain:

$$(2.14) \quad \hat{\Phi}_{ij}(\mathbf{k}) \hat{u}_j(\mathbf{k}) = -ik_i \hat{\Psi}(\mathbf{k}),$$

the solution of which is

$$(2.15) \quad \hat{u} = -\frac{1}{\lambda + 2\mu} \frac{ik_i}{k^2} e^{-k/4\gamma^2},$$

where

$$(2.16) \quad \frac{1}{\gamma^2} = \frac{1}{\alpha^2} - \frac{1}{\beta^2}.$$

If  $\alpha < \beta$  — i.e., the forces are diffused over a distance greater than the range of interactions — then  $\gamma^2 > 0$ , and there exists a smooth solution which, by retransformation of (2.15), is equal to

$$(2.17) \quad u_i(\mathbf{x}) = -\frac{1}{4\pi} \frac{1}{\lambda + 2\mu} \partial_i \frac{\text{erf}(\gamma r)}{r},$$

where erf denotes the corresponding error function (see e.g., LUKE, 1969). If  $\alpha = \beta$ , there exists a singular solution

$$(2.18) \quad u_i(\mathbf{x}) = -\frac{1}{4\pi} \frac{1}{\lambda + 2\mu} \partial_i \frac{1}{r}$$

which coincides with the classical solution corresponding to  $\psi(\mathbf{x}) = \delta^{(3)}(\mathbf{x})$ .

In the case of  $\alpha > \beta$ , there is no solution in the class of tempered distributions. There exists, in fact, a solution given by the Eq. (2.17) with imaginary  $\gamma$  derived from (2.16). It can be checked by direct computation, the integral being, provided that  $\alpha^{-1} \neq 0$ , very well convergent. This solution, however, grows up exponentially at infinity.

This is not what can be expected on physical grounds. Although a good solution exists when the forces are sufficiently diffused, the necessary degree of diffusion is determined

by the range of interactions instead of by the interatomic distance. The range  $\beta^{-1}$  can in principle be made very large so that the inequalities

$$(2.19) \quad a \ll \alpha^{-1} < \beta$$

can be satisfied very well. In spite of that, no acceptable solution exists in this case.

What has been said in this paragraph refers directly to a particular case of an integral equation. Nevertheless, it shows that in formulating nonlocal continuum theories, due attention to the mathematical side of the problem is necessary.

### 3. Nonlocal fundamental equations

Now, we shall try to investigate nonlocal theories of continuous elastic media in a slightly more systematic way. The very first question we meet here concerns the kind of governing equation we should choose. The almost automatic answer that it is an integral equation is in many respects not satisfactory. From the mathematical point of view, such an answer tells us almost nothing, unless we specify in what sense the integrals involved are to be understood. Classical integrals are usually too restrictive, since many singular functions of physical interest cannot be integrated in a classical way. Even if we choose some generalized notion of the integral, we cannot guarantee that a non-differential equation, if acceptable on physical grounds, has to be an integral one or, at least, can be reasonably written by means of such integrals.

The whole question is not unimportant, because the governing equation can forejudge physically important features of its solutions. Bearing this in mind, we shall discuss a wide class of linear governing equations which, apart from restrictions of direct physical meaning, we submit to some mathematical assumptions of rather general character only.

More specifically, we are going to investigate equations of the form

$$(3.1) \quad L\mathbf{u} = \mathbf{f}$$

or, in index notation,

$$(3.1') \quad L_{ij}u_i = f_j,$$

where  $\mathbf{u}$  and  $\mathbf{f}$  are the displacement and force fields, respectively, and  $L$  is a certain linear operator. Where the fields  $\mathbf{u}$  and  $\mathbf{f}$  are concerned we shall consider them tempered distributions on the three-dimensional Euclidean space. The operator  $L$  will, as a rule, be defined for a certain class  $U$  of tempered distributions, not necessarily for all of them. This class will depend on  $L$ , so we do not specify it in advance.

The basic assumptions on  $L$  and  $U$  are the following:

a. The operator  $L$  is continuous on  $U$ . The continuity we assume here is a sequential one in the following sense: whenever a sequence  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots \in U$  is convergent to  $\mathbf{u} \in U$ ,

$$(3.2) \quad \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots \rightarrow \mathbf{u},$$

then

$$(3.3) \quad \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \dots \rightarrow \mathbf{f},$$

the arrows indicating the convergence in the space of distributions. The symbols  $\mathbf{f}, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \dots$  denote the force fields that correspond to the displacement fields  $\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots$  by the Eq. (3.1).

b. The medium is homogeneous. In order to express precisely the homogeneity assumption, we make use of the translation operator  $T_{\mathbf{c}}$  whose action on an arbitrary field consists in translating it by a constant vector  $\mathbf{c}$ , e.g.:

$$(3.4) \quad T_{\mathbf{c}}\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x} - \mathbf{c}).$$

We say that the medium is homogeneous if the operator  $L$  commutes with the translation operators,

$$(3.5) \quad LT_{\mathbf{c}} = T_{\mathbf{c}}L,$$

for an arbitrary  $\mathbf{c}$ . Thus, the above implies that if  $\mathbf{u} \in U$ , then  $T_{\mathbf{c}}\mathbf{u} \in U$ .

For the sake of simplicity, we assume also that the medium is centrosymmetric.

This assumption can be expressed in a form similar to (3.5) by writing the inversion operator in place of  $T_{\mathbf{c}}$ .

c) The class  $U$ , on which the operator  $L$  is defined, contains all the functions of the form

$$(3.6) \quad \mathbf{u}(\mathbf{x}) = \text{Re} \mathbf{a} e^{i\mathbf{k}\mathbf{x}}$$

with arbitrary real wave vectors  $\mathbf{k}$  and complex amplitudes  $\mathbf{a}$ .

The above assumptions determine the general form of the operator  $L$ . The fields  $\mathbf{u}$  and  $\mathbf{f}$  are Fourier-transformable into some tempered distributions  $\hat{\mathbf{u}}$  and  $\hat{\mathbf{f}}$ , so that the Eq. (3.1) can be equivalently written in the form

$$(3.7) \quad \hat{L}\hat{\mathbf{u}} = \hat{\mathbf{f}},$$

where  $\hat{L}$  is another linear operator. The operators  $L$  and  $\hat{L}$  uniquely determine each other.

On the other hand, the homogeneity assumption implies that, for  $\mathbf{u}$  given by the Eq. (3.6),  $L\mathbf{u}$  must be of the form:

$$(3.8) \quad L\mathbf{u} = \text{Re} \mathbf{b} e^{i\mathbf{k}\mathbf{x}}$$

with another amplitude  $\mathbf{b}$ . This amplitude, in turn, must depend on  $\mathbf{a}$  linearly, so that

$$(3.9) \quad b_i = A_{ij}(\mathbf{k}) a_j,$$

the matrix  $A_{ij}$ , together with its dependence on  $\mathbf{k}$ , being completely determined by the operator  $L$ .

Now, we make use of continuity condition. This we do in three steps:

1. Consider a sequence  $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \dots$  which converges to a certain wave vector  $\mathbf{k}$ . Then

$$(3.10) \quad e^{i\mathbf{k}_1\mathbf{x}}, e^{i\mathbf{k}_2\mathbf{x}}, e^{i\mathbf{k}_3\mathbf{x}}, \dots \rightarrow e^{i\mathbf{k}\mathbf{x}},$$

and, because of the continuity of  $L$ ,

$$(3.11) \quad A_{ij}(\mathbf{k}_1), A_{ij}(\mathbf{k}_2), A_{ij}(\mathbf{k}_3), \dots \rightarrow A_{ij}(\mathbf{k}).$$

Thus the matrix  $A_{ij}$  is a continuous function of  $\mathbf{k}$ .

2. Consider  $\mathbf{u}(\mathbf{x})$  of the form

$$(3.12) \quad \mathbf{u}(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3k e^{i\mathbf{k}\cdot\mathbf{x}} \hat{\mathbf{u}}(\mathbf{k}),$$

where  $\hat{\mathbf{u}}(\mathbf{k})$  is a continuous function of bounded support in the  $\mathbf{k}$  space.

In this case,  $\mathbf{u}(\mathbf{x})$  is given by a Riemann integral which, by definition, is a limit of finite sums. These sums are finite linear combinations of  $e^{i\mathbf{k}_m\cdot\mathbf{x}}$  with different  $\mathbf{k}_m$ 's and converge to  $\mathbf{u}(\mathbf{x})$ . Hence, if  $\mathbf{u}(\mathbf{x}) \in U$ , then, by the continuity of  $L$ ,

$$(3.13) \quad (\hat{L}\hat{\mathbf{u}})_i = A_{ij}(\mathbf{k})\hat{u}_j(\mathbf{k}),$$

— i.e., the operator  $\hat{L}$  acts as multiplication by the matrix function  $A_{ij}(\mathbf{k})$ .

It follows, in particular, that all the fields  $\mathbf{u}$  whose Fourier transform  $\hat{\mathbf{u}}$  are continuous and of bounded support can be included into  $U$ , which we assume to be done.

3. Consider an arbitrary  $\mathbf{u} \in U$ . Since any distribution is a limit of a sequence of continuous functions with compact supports, so is  $\hat{\mathbf{u}}$ , the Fourier transform of  $\mathbf{u}$ . Therefore, again by the continuity of  $L$ , the extension of this operator from the functions specified in 2 onto  $U$  is unique.

Hence we arrive at the conclusion that the matrix function  $A_{ij}(\mathbf{k})$  determines the operator  $L$  on  $U$  uniquely.

Thus, under the assumptions a, b and c, the general form of the Eq. (3.7) is

$$(3.14) \quad A_{ij}(\mathbf{k})\hat{u}_j(\mathbf{k}) = \hat{f}_i(\mathbf{J}),$$

where  $A_{ij}(\mathbf{k})$  has to be a continuous function of  $\mathbf{k}$ .

The distributions  $\hat{\mathbf{u}}(\mathbf{k})$  and  $\hat{\mathbf{f}}(\mathbf{k})$ , being the Fourier transforms of real distributions  $\mathbf{u}(\mathbf{x})$  and  $\mathbf{f}(\mathbf{x})$ , must satisfy the following relations:

$$(3.15) \quad \hat{\mathbf{u}}^*(\mathbf{k}) = \hat{\mathbf{u}}(-\mathbf{k}), \quad \hat{\mathbf{f}}^*(\mathbf{k}) = \hat{\mathbf{f}}(-\mathbf{k}).$$

Hence

$$(3.16) \quad A_{ij}^*(\mathbf{k}) = A_{ij}(-\mathbf{k})$$

and, because of central symmetry of the medium,

$$(3.17) \quad A_{ij}(-\mathbf{k}) = A_{ij}(\mathbf{k}).$$

Thus  $A_{ij}(\mathbf{k})$  is a real and even function of  $\mathbf{k}$ .

#### 4. Energy and stability

The expression for the total deformation energy of a nonlocal elastic medium can be derived from the form of the governing equation. The energy corresponding to displacements  $\mathbf{u}(\mathbf{x})$  produced by forces  $\mathbf{f}(\mathbf{x})$  equals

$$(4.1) \quad W = \frac{1}{2} \int d^3x \mathbf{u}\mathbf{f} = \frac{1}{2} \int d\mathbf{x} \mathbf{u} L \mathbf{u},$$

which follows from integrating the elementary work

$$(4.2) \quad \delta W = \int d^3x \delta\mathbf{u}\mathbf{f},$$

making use of linearity of the Eq. (1.1). In Fourier representation, the expression (4.1) can be written as

$$(4.3) \quad W = \frac{1}{2} \frac{1}{(2\pi)^3} \int d^3k \hat{u}_i^*(\mathbf{k}) A_{ij}(\mathbf{k}) \hat{u}_j(\mathbf{k}).$$

Now, consider a cyclic deformation process of the form

$$\begin{array}{c} \mathbf{u}^{(2)}(\mathbf{x}) \\ \swarrow \quad \searrow \\ \mathbf{u}^{(1)}(\mathbf{x}) \leftarrow 0 \end{array}$$

with some displacement fields  $\mathbf{u}^{(1)}(\mathbf{x})$ ,  $\mathbf{u}^{(2)}(\mathbf{x})$  and the corresponding force fields  $\mathbf{f}^{(1)}(\mathbf{x})$ ,  $\mathbf{f}^{(2)}(\mathbf{x})$ . The medium being elastic, the work done in this process has to be zero:

$$(4.4) \quad 0 = W_{01} + W_{12} + W_{20} = \frac{1}{2} \int d^3x (\mathbf{u}^{(2)} \mathbf{f}^{(1)} - \mathbf{u}^{(1)} \mathbf{f}^{(2)}).$$

On transforming this relation to Fourier representation and making use of the Eq. (3.14), we obtain:

$$(4.5) \quad \int d^3k \hat{u}_i^*(\mathbf{k}) [A_{ij}(\mathbf{k}) - A_{ij}^*(\mathbf{k})] \hat{u}_i(\mathbf{k}) = 0.$$

Let us note that the expression (4.3) is well defined for sufficiently many  $\hat{\mathbf{u}}(\mathbf{k})$ : at least for all the continuous functions of bounded support. Therefore, from the relation (4.5) it follows that

$$(4.6) \quad A_{ij}(\mathbf{k}) = A_{ij}^*(\mathbf{k}).$$

Taking into account the Eqs. (3.16) and (3.17), we have then

$$(4.7) \quad A_{ij}(\mathbf{k}) = A_{ij}(-\mathbf{k}) = A_{ij}^*(\mathbf{k}) = A_{ij}(\mathbf{k}).$$

Moreover, we assume the medium to be stable. According to KUNIN (1968), the stability condition requires the roots  $\omega_1^2(\mathbf{k})$ ,  $\omega_2^2(\mathbf{k})$ ,  $\omega_3^2(\mathbf{k})$  of the characteristic equation

$$(4.8) \quad \det(A_{ij}(\mathbf{k}) - \omega^2 \delta_{ij}) = 0$$

to be positive for any real  $\mathbf{k} \neq 0$ . Thus the matrix  $A_{ij}(\mathbf{k})$  must be positive definite for  $\mathbf{k} \neq 0$  and, in particular,

$$(4.9) \quad \det(A_{ij}(\mathbf{k})) \neq 0 \quad \text{for} \quad \mathbf{k} \neq 0.$$

So far we have made no assumptions concerning the relation between nonlocal and classical elasticity. We assume that the Eq. (3.14) agrees with its classical counterpart in the limit  $\mathbf{k} \rightarrow 0$ . Thus

$$(4.10) \quad A_{ij}(\mathbf{k}) = c_{ijm} k_i k_m + o(k^2) \quad \text{when} \quad k \rightarrow 0.$$

This completes the list of assumptions concerning the operator  $L$ .

## 5. The singular order of the operator $L$

Let  $p$  be a real number. We define the following quantity

$$(5.1) \quad \|L\|_p = \frac{1}{(2\pi)^3} \int d^3k (1+k^2)^{-p/2} \text{tr} A(\mathbf{k}),$$

where

$$(5.2) \quad \text{tr} A(\mathbf{k}) = A_{11}(\mathbf{k}) + A_{22}(\mathbf{k}) + A_{23}(\mathbf{k}) = \omega_1^2(\mathbf{k}) + \omega_2^2(\mathbf{k}) + \omega_3^2(\mathbf{k}) \geq 0.$$

From the definition (5.1) it follows immediately that

$$(5.3) \quad \|L\|_{p'} < \|L\|_p \quad \text{for } p' > p.$$

Therefore, the set of numbers  $p$ , for which the inequality

$$(5.4) \quad \|L\|_p < \infty$$

holds, can conveniently be characterized by the quantity  $s(L)$  defined as

$$(5.5) \quad s(L) = \inf p: \|L\|_p < \infty.$$

With that we understand that, if  $\|L\|_p < \infty$  for all real  $p$ , then  $s(L) = -\infty$ . In the case in which  $\|L\|_p = \infty$  for all real  $p$ , we define  $s(L) = +\infty$ . The quantity  $s(L)$  will be called the singular order of the operator  $L$ .

If  $s(L) = s$ , where  $s$  is finite, then either

$$(5.6) \quad \|L\|_p \begin{cases} = \infty & \text{for } p \leq s, \\ < \infty & \text{for } p > s, \end{cases}$$

or

$$(5.7) \quad \|L\|_p \begin{cases} = \infty & \text{for } p < s, \\ < \infty & \text{for } p \geq s. \end{cases}$$

If we want to stress the difference, we shall say that the singular order of the operator  $L$  is "exactly  $s$ " in the first case, and "almost  $s$ " in the second.

By definition,

$$(5.8) \quad \text{almost } s < \text{exactly } s.$$

One can easily observe that, if  $L = L' + L''$  and  $s(L') > s(L'')$ , then

$$(5.9) \quad s(L) = s(L').$$

If  $L$  is a (positive definite) differential operator of order  $m$ , then

$$(5.10) \quad s(L) = \text{exactly } m + 3.$$

Hence the singular order can be regarded as a generalization of the order of differential operators, shifted by 3 for convenience.

## 6. The convolution equations

Consider first the case in which the matrix function is a tempered distribution (i.e. all its components are tempered distributions). Then we have

$$(6.1) \quad A_{ij}(\mathbf{k}) = \hat{\Phi}_{ij}(\mathbf{k}),$$

where  $\hat{\Phi}_{ij}(\mathbf{k})$  is the Fourier transform of a tempered distribution  $\Phi_{ij}(\mathbf{x})$ .

The fundamental equation (3.1') can now be written in the convolution form

$$(6.2) \quad \Phi_{ij} * u_j = F_i,$$

with the kernel  $\Phi_{ij}(\mathbf{x})$ . The class  $u_j(\mathbf{x})$ , for which this equation is defined, still depends on a particular form of  $\Phi_{ij}$ . In any case, we consider the Eq. (6.2) equivalent to the Eq. (3.14) with  $A_{ij}(\mathbf{k})$  given by (6.1).

We have the following

**THEOREM 6.1.** *In order that  $A_{ij}(\mathbf{k})$  be a tempered distribution it is necessary and sufficient that*

$$(6.3) \quad s(L) < +\infty.$$

**P r o o f.** If  $A_{ij}(\mathbf{k})$  is a tempered distribution, then, by (5.2),  $\text{tr} A(\mathbf{k})$  is a positive tempered distribution. According to GELFAND and VILENKIN (1964), such a distribution is given by a tempered measure, which implies inequality (5.4) for a certain real  $p$ . In consequence, we have inequality (6.3). To prove the converse, let us note that, the matrix  $A_{ij}(\mathbf{k})$  being at least positive semi-definite for all  $\mathbf{k}$ , the inequality

$$(6.4) \quad |A_{ij}(\mathbf{k})| \leq 2 \text{tr} A(\mathbf{k})$$

holds for any pair of indices  $i, j$ . Thus, if the condition (6.3) is satisfied, then there exists a real  $p$  such that

$$(6.5) \quad \int d^3k (1+k^2)^{-p/2} |A_{ij}(\mathbf{k})| \leq 2 \|L\|_p < \infty$$

which shows that  $A_{ij}(\mathbf{k})$  is a tempered distribution.

The singular order of a convolution equation provides a measure of the singularity of the kernel  $\Phi_{ij}$ . The following theorems reveal the corresponding relation.

**THEOREM 6.2.** *The kernel  $\Phi_{ij}(\mathbf{x})$  is a continuous function, if and only if,*

$$(6.6) \quad s(L) \leq \text{almost } 0.$$

**P r o o f.** If the inequality (6.6) is satisfied, then the inequality (6.5) holds for  $p = 0$  — i.e. the function  $A_{ij}(\mathbf{k})$  is summable. By the Riemann-Lebesgue theorem, the kernel  $\Phi_{ij}(\mathbf{x})$ , which is the Fourier transform of  $A_{ij}(\mathbf{k})$ , is continuous. Conversely, if  $\Phi_{ij}(\mathbf{x})$  is a continuous function, then

$$(6.7) \quad \text{tr} \Phi = \Phi_{11}(\mathbf{x}) + \Phi_{22}(\mathbf{x}) + \Phi_{33}(\mathbf{x})$$

is a positive definite continuous function and, according to Bochner's theorem (GELFAND and VILENKIN, 1964), its Fourier transform is given by a finite measure.

This implies the inequality (5.4) for  $p = 0$  and, in consequence, the condition (6.6).

**THEOREM 6.3.** *If*

$$(6.8) \quad s(L) \leq \text{almost } -m,$$

*where  $m$  is a positive integer, then  $\Phi(\mathbf{x})$  has continuous derivatives up to order  $m$ .*

**P r o o f.** Condition (6.8) implies the inequality (6.5) for  $p = -m$ . Taking into account the inequality

$$(6.9) \quad |k^\mu| \leq (1+k^2)^{m/2} \quad \text{for } |\mu| = m,$$

we conclude that the derivatives  $\partial^\mu \Phi_{ij}(\mathbf{x})$  with  $|\mu| \leq m$  have summable Fourier transforms, and therefore, by the Riemann-Lebesgue theorem, are continuous.

**Remark.** For even integers  $m$ , the theorem converse to (6.3) is true. This can be shown by considering the kernel

$$(6.10) \quad \Phi'_{ij} = (1 - \Delta)^{m/2} \Phi_{ij},$$

which defines an admissible operator  $L'$  with the matrix  $A'_{ij}(\mathbf{k})$  given by

$$(6.11) \quad A'_{ij}(\mathbf{k}) = (1 + k^2)^{m/2} A_{ij}(\mathbf{k}).$$

If the derivatives  $\partial^\mu \Phi_{ij}$  are continuous for  $|\mu| \leq m$ , then the kernel  $\Phi'_{ij}$  is continuous and, by Theorem 6.2,

$$(6.12) \quad s(L) = s(L') - m \leq \text{almost } -m.$$

From this remark and from Theorem 6.3, the following theorem follows immediately:

**THEOREM 6.4.** *The kernel  $\Phi_{ij}(\mathbf{x})$  is an infinitely differentiable function, if and only if,*

$$(6.13) \quad s(L) = -\infty.$$

Moreover, we have

**THEOREM 6.5.** *If  $\text{tr} \Phi(\mathbf{x})$  is a bounded function in a certain neighbourhood of  $\mathbf{x} = 0$ , then the kernel  $\Phi_{ij}(\mathbf{x})$  is a continuous function everywhere.*

**Proof.** If the assumption of the theorem is satisfied, then  $\text{tr} \Phi(\mathbf{x})$  can be represented in the form:

$$(6.14) \quad \text{tr} \Phi = f_1 + f_2,$$

where  $f_1$  is a bounded function,

$$(6.15) \quad |f_1(\mathbf{x})| < C$$

and  $f_2$  is a tempered distribution such that

$$(6.16) \quad f_2(\mathbf{x}) = 0 \quad \text{for } |\mathbf{x}| \leq \varepsilon,$$

where  $C$  and  $\varepsilon$  are certain positive constants. Let  $\psi_\alpha$  denote the function defined by the Eq. (2.12) for a certain value of the parameter  $\alpha$ . Then

$$(6.17) \quad |(f_1, \psi_\alpha)| \leq C$$

and there exists a polynomial  $P(\alpha) > 0$  such that

$$(6.18) \quad |(f_2, \psi_\alpha)| \leq e^{-\alpha^2 \varepsilon^2} P(\alpha_\alpha).$$

Thus

$$(6.19) \quad \limsup_{\alpha \rightarrow \infty} |(\text{tr} \Phi, \psi_\alpha)| \leq C.$$

On the other hand,

$$(6.20) \quad (\text{tr} \Phi, \psi_\alpha) = (\text{tr} A, \hat{\psi}_\alpha) = \frac{1}{(2\pi)^3} \int d^3 k \text{tr} A(\mathbf{k}) e^{-k^2/4\alpha^2}$$

and

$$(6.21) \quad \lim_{\alpha \rightarrow \infty} (\text{tr} \Phi, \psi_\alpha) = \|L\|_0 \leq C,$$

where the last inequality follows from (6.19). Thus  $s(L) \leq \text{almost } 0$  and, by Theorem 6.2, the kernel  $\Phi_{ij}(\mathbf{x})$  is continuous.

THEOREM 6.6. *If*

$$(6.20) \quad s(L) \leq \text{almost } m,$$

where  $m$  is a positive integer, then the kernel  $\Phi_{ij}(\mathbf{x})$  can be expressed as a finite sum of continuous functions and their derivatives of order not greater than  $m$ .

PROOF. We shall prove this theorem by the construction of corresponding representations of the kernel  $\Phi_{ij}$ .

Let

$$(6.21) \quad q = \begin{cases} \frac{m}{2} & \text{for even } m \\ \frac{2}{m+1} & \text{for odd } m. \end{cases}$$

Let  $\Psi_{ij}(\mathbf{x})$  be a tempered distribution whose Fourier transform is

$$(6.22) \quad \hat{\Psi}_{ij}(\mathbf{x}) = (1+k^2)^{-q} A_{ij}(\mathbf{k}).$$

The distribution  $\Psi_{ij}(\mathbf{x})$  represents the kernel of an admissible convolution operator  $M$  of singular order

$$(6.23) \quad s(M) = s(L) + q \leq \text{almost } m - 2q.$$

If  $m$  is even, then by Theorem 6.2 the kernel  $\Psi_{ij}(\mathbf{x})$  is continuous and

$$(6.24) \quad \Phi_{ij} = (1-\Delta)^q \Psi_{ij}$$

is a representation of the desired form. If  $m$  is odd, then by Theorem 6.3 the kernel  $\Psi_{ij}$  is continuously differentiable, so that  $\Psi_{ij}$  and

$$(6.25) \quad \chi_{ij} = \partial_l \psi_{ij}$$

are continuous functions. Hence the representation we are looking for can be written as

$$(6.26) \quad \Phi_{ij} = (1-\Delta)^{q-1} \partial_l \chi_{ij} + (1-\Delta)^{q-1} \Psi_{ij}.$$

It follows from Theorem 6.5 that whenever the singular order of a convolution operator is negative (or exactly 0), i.e.,

$$(6.21) \quad s(L) \leq \text{exactly } 0,$$

then at  $\mathbf{x} = 0$  the corresponding kernel has a singularity which cannot be represented by a bounded function.

This singularity, however, can be represented by derivatives of continuous functions, and Theorem 6.6 gives the dependence between the singular order of the operator and the necessary order of these derivatives.

The following Table 1 gives a few simple examples of singularities of admissible kernels at  $\mathbf{x} = 0$ , and specifies the singular order of the corresponding operators.

Table 1.

|        | $0 < \alpha \neq 2, 4, \dots$<br>$r^\alpha$ | $-\log r$ | $r^{-\alpha}$<br>$\alpha > 0$ | $\delta^{(3)}(\mathbf{x})$ | $-\Delta^{(3)}(\mathbf{x})$ |
|--------|---|-----------|-------------------------------|----------------------------|-----------------------------|
| $s(L)$ | $-\alpha$                                   | 0         | $\alpha$                      | 3                          | 5                           |

All the singular orders listed in this Table are of the "exactly" type.

### 7. The fundamental solution and the singular hardness of an elastic material

Consider now the inverse matrix  $A_{ij}^{-1}(\mathbf{k})$ . According to (4.9), this matrix is well defined at any  $\mathbf{k} \neq 0$ . Considered as a function of  $\mathbf{k}$ , this matrix is continuous for  $\mathbf{k} \neq 0$  and has a singularity at  $\mathbf{k} = 0$ .

However, as follows from the condition (4.10), this singularity is summable. Hence  $A_{ij}(\mathbf{k})$  uniquely defines a locally summable function on the  $\mathbf{k}$ -space, and it will be understood in this sense.

By the equation

$$(7.1) \quad \hat{u}_i(\mathbf{k}) = A_{ij}^{-1}(\mathbf{k}) \hat{f}_j(\mathbf{k}),$$

the function  $A_{ij}^{-1}(\mathbf{k})$  defines an operator from  $L[U]$  into  $U$  which will be denoted by  $L^{-1}$ :

$$(7.2) \quad L_{ij}^{-1} f_j(\mathbf{x}) = u_i(\mathbf{x}).$$

In the case in which  $A_{ij}^{-1}(\mathbf{k})$  is a tempered distribution (and this depends solely on its asymptotic behaviour for  $k \rightarrow \infty$ ), there exists a tempered distribution  $G_{ij}(\mathbf{x})$  whose Fourier transform

$$(7.3) \quad \hat{G}_{ij}(\mathbf{k}) = A_{ij}^{-1}(\mathbf{k}).$$

In this case, the distribution  $G_{ij}(\mathbf{x})$  satisfies the equation

$$(7.4) \quad L_{ij} G_{jm}(\mathbf{x}) = \delta_{im} \delta^{(3)}(\mathbf{x})$$

and will be called the fundamental solution.

The function  $A_{ij}^{-1}(\mathbf{k})$  being locally integrable, the definitions (5.1) and (5.5) make sense for the operator  $L^{-1}$ . Moreover, all the theorems of paragraph 6 apply to the operator  $L^{-1}$ , provided that the following substitutions are made:

$$(7.5) \quad \begin{aligned} \Phi_{ij} &\rightarrow G_{ij}, \\ A_{ij} &\rightarrow A_{ij}^{-1}, \\ s(L) &\rightarrow s(L^{-1}). \end{aligned}$$

In particular, the fundamental solution

- a exists,
- b is continuous (bounded),
- c is infinitely differentiable.

if and only if,

$$(7.6) \quad \begin{aligned} \text{a} \quad & s(L^{-1}) < +\infty, \\ \text{b} \quad & s(L^{-1}) \leq \text{almost } 0, \\ \text{c} \quad & s(L^{-1}) = -\infty, \end{aligned}$$

respectively.

The quantity

$$(7.7) \quad h(L) = -s(L^{-1})$$

will be called the singular hardness of the corresponding elastic material.

The idea here is that if the material is "singular hard", the singularity of the displacement field created by applying a concentrated force is weak. And if the material is "singular soft", a concentrated force creates a strong singularity in the displacement field. The quantity (7.7) provides a numerical measure of this property.

### 8. The relation between $s(L)$ and $s(L^{-1})$

Now, we shall prove the following fundamental inequality between the singular orders of the operators  $L$  and  $L^{-1}$ :

$$(8.1) \quad s(L^{-1}) \geq 6 - s(L).$$

The proof is based on the following inequality:

$$(8.2) \quad \int \operatorname{tr}(A^2)(1+k^2)^{-p/2} d^3k \int \operatorname{tr}(B^2)(1+k^2)^{-p'/2} d^3k \geq \left( \int |\operatorname{tr}AB|(1+k^2)^{-\frac{p+p'}{4}} d^3k \right)^2,$$

which can be obtained from Schwartz's inequalities for traces and integrals and is valid for any measurable matrix functions  $A_{ij}(\mathbf{k})$ ,  $B_{ij}(\mathbf{k})$  and arbitrary real numbers  $p$ ,  $p'$ .

By substituting  $A = A^{1/2}$  and  $B = A^{-1/2}$  into (8.2), we obtain

$$(8.3) \quad \|L\|_p \|L^{-1}\|_{p'} \geq 9 \|1\|_{\frac{p+p'}{2}}^2.$$

Thus, whenever

$$(8.4) \quad \|L\|_p < \infty \quad \text{and} \quad \|L^{-1}\|_{p'} < \infty,$$

then

$$(8.5) \quad p' > 6 - p,$$

the right-hand side of the inequality (8.3) being, on the contrary, divergent. Hence

$$(8.6) \quad \inf p' \geq \sup(6 - p) = 6 - \inf p,$$

which proves the inequality (8.1).

By introducing the following convention:

$$(8.7) \quad \text{almost } s = \text{exactly } s,$$

the validity of (8.1) is extended to singular orders labelled as "almost" or "exactly".

**COROLLARY 8.1.** *For the fundamental solution  $G_{ij}$  to be bounded (continuous), it is necessary that*

$$(8.8) \quad s(L) \geq \text{exactly } 6.$$

**COROLLARY 8.2.** *If the kernel  $\Phi_{ij}$  is bounded (continuous), then*

$$(8.9) \quad s(L^{-1}) \geq \text{exactly } 6.$$

**COROLLARY 8.3.** *If the kernel  $\Phi_{ij}$  is infinitely differentiable, then the fundamental solution does not exist.*

The last corollary explains the failure of the example discussed in paragraph 2: the kernel  $\Phi_{ij}$  defined by the Eqs. (2.1)–(2.3) is infinitely differentiable.

Generally, the smoother the kernel  $\Phi_{ij}$  of a convolution equation, the more singular

must be the fundamental solution  $G_{ij}$ . The weakest possible singularity of  $G_{ij}$  corresponds to the value of  $s(L^{-1})$  given by the Eq:

$$(8.10) \quad s(L^{-1}) = 6 - s(L).$$

The following two examples illustrate how inequality (8.1) works in some particular cases.

a. Consider an isotropic medium described by a convolution fundamental equation. In that case, the general form of the kernel  $\Phi_{ij}(\mathbf{x})$  is

$$(8.11) \quad \Phi_{ij}(\mathbf{x}) = -(\Delta\delta_{ij} - \partial_i\partial_j)\Psi_1(r) - \partial_i\partial_j\Psi_2(r),$$

where  $\Psi_1(r)$  and  $\Psi_2(r)$  are spherically symmetric tempered distributions.

Let  $\Psi_1(r)$  and  $\Psi_2(r)$  have singularities at  $r = 0$  only, and let these singularities have the form:

$$(8.12) \quad \Psi_1(r) \sim r^{2-\alpha_1}, \quad \Psi_2(r) \sim r^{2-\alpha_2}, \quad \text{for } r \rightarrow 0$$

with some non-integer  $\alpha_1$  and  $\alpha_2$ . Then the singular order of the corresponding operator  $L$  equals

$$(8.13) \quad s(L) = \text{exactly } \max(\alpha_1, \alpha_2).$$

The fundamental solution has the same form:

$$(8.14) \quad G_{ij}(\mathbf{x}) = -(\Delta\delta_{ij} - \partial_i\partial_j)H_1(r) - \partial_i\partial_jH_2(r)$$

and, as inspection of the corresponding Fourier transforms shows, the strongest singularity is again at  $r = 0$ , and

$$(8.15) \quad H_1(r) \sim r^{\alpha_1-4}, \quad H_2(r) \sim r^{\alpha_2-4}, \quad \text{for } r \rightarrow 0.$$

Hence

$$(8.16) \quad s(L^{-1}) = \text{exactly } 6 - \min(\alpha_1, \alpha_2).$$

Thus the present example Eq. (8.10) holds numerically if  $\alpha_1 = \alpha_2$ . On the contrary, the sharp numerical inequality in (8.10) is valid.

b. Consider an elastic medium described by a nonlocal stress-strain relation of the form proposed by KRÖNER (1967):

$$(8.17) \quad \sigma_{ij}(\mathbf{x}) = c_{ijkl}\varepsilon_{kl}(\mathbf{x}) + \int c_{ijkl}^*(\mathbf{x}-\mathbf{x}')\varepsilon_{kl}(\mathbf{x}')d^3x'.$$

The corresponding fundamental equation has the convolution form (6.2) with the kernel

$$(8.18) \quad \Phi_{ij}(\mathbf{x}) = -\partial_k\partial_l(c_{ikjl}\delta^{(3)}(\mathbf{x}) + c_{ikjl}^*(\mathbf{x})).$$

Let the function  $c_{ikjl}^*(\mathbf{x})$  be absolutely integrable. Then, by writing the corresponding Fourier transforms and making use of the Riemann-Lebesgue theorem, we obtain

$$(8.19) \quad s(L) = \text{exactly } 5.$$

Hence, according to Corollary 8.1, the fundamental solution cannot be bounded or continuous. In fact, an inspection of relevant Fourier transforms shows that

$$(8.20) \quad s(L^{-1}) = \text{exactly } 1.$$

Thus, in the case considered, the Eq. (8.10) is numerically valid. This also refers to the classical case ( $c_{ijkl}^* = 0$ ).

### 9. The non-convolution equations

In paragraph 6, we have discussed the case in which  $A_{ij}(\mathbf{k})$  is a tempered distribution. Now, we shall consider the remaining case. If  $A_{ij}(\mathbf{k})$  is not a tempered distribution, then there is no distribution, even non-tempered, which would allow the fundamental equation to be written in the convolution form (6.2). Nevertheless, such a function  $A_{ij}(\mathbf{k})$  defines uniquely an operator  $L$  in the Eq. (3.1), and this operator has all the properties required. We shall refer to this case as the non-convolution case. The corresponding singular order of  $L$  is

$$(9.1) \quad s(L) = +\infty$$

In this case, the inequality (8.1) does not restrict the regularity of the fundamental solution, which can be an infinitely differentiable function.

In fact, we have

PROPOSITION 9.1. If for any real  $m$  the inequalities

$$(9.2) \quad \omega_1^2(\mathbf{k}) > k^m, \quad \omega_2^2(\mathbf{k}) > k^m, \quad \omega_3^2(\mathbf{k}) > k^m$$

are satisfied, provided that the vector  $\mathbf{k}$  is sufficiently large, then

$$(9.3) \quad s(L) = +\infty, \quad s(L^{-1}) = -\infty.$$

This proposition follows directly from the definition of  $s(L)$  and  $s(L^{-1})$ .

Consider an example. Let

$$(9.4) \quad A_{ij}(\mathbf{k}) = [\mu k^2 \delta_{ij} + (\lambda + \mu) k_i k_j] e^{k^2/4\beta^2}.$$

It has a form similar to (2.8) but with a positive exponent. The Fourier transform of the fundamental solution, as (2.9), is

$$(9.5) \quad \hat{G}_{ij}(\mathbf{k}) = \left[ \frac{1}{\mu} \left( \frac{\delta_{ij}}{k^2} - \frac{k_i k_j}{k^4} \right) + \frac{1}{\lambda + 2\mu} \frac{k_i k_j}{k^4} \right] e^{-k^2/4\beta^2}.$$

Hence the fundamental solution equals

$$(9.6) \quad G_{ij}(\mathbf{x}) = -\frac{1}{4\pi\mu} \delta_{ij} \frac{\text{erf}(\beta r)}{r} - \frac{1}{4\pi} \left( \frac{1}{\mu} - \frac{1}{\lambda + 2\mu} \right) \partial_i \partial_j \left[ \frac{1}{r} \int_0^r (r-s) \text{erf}(\beta s) ds \right].$$

This solution is not only infinitely differentiable: it is an entire analytic function of  $\mathbf{x}$ .

### 10. The energy of force centres

Consider a concentrated force of the Dirac  $\delta$ -type:

$$(10.1) \quad f_i(\mathbf{x}) = F_i \delta^{(3)}(\mathbf{x}).$$

The corresponding deformation energy equals

$$(10.2) \quad W = \frac{1}{2} G_{ij}(0) F_i F_j,$$

where  $G_{ij}(\mathbf{x})$  is the fundamental solution. Thus, if  $s(L^{-1}) \leq \text{almost } 0$ , this energy is finite since the fundamental solution is continuous.

If  $s(L^{-1}) \geq \text{exactly } 0$ , the energy (10.2) can be finite for some particular directions of the vector  $F_i$ , but not for all of them. In order to show that, let us consider the average energy of three mutually perpendicular unit forces  $F_i^{(1)}, F_i^{(2)}, F_i^{(3)}$ :

$$(10.3) \quad \bar{W}_0 = -\frac{1}{6} G_{ij}(0) \sum_{\alpha=1}^3 F_i^{(\alpha)} F_j^{(\alpha)} = -\frac{1}{6} G_{ii}(0).$$

Making use of the Fourier representation, we obtain

$$(10.4) \quad \|L^{-1}\|_0 = 6 \bar{W}_0.$$

Thus, if  $s(L^{-1}) \geq \text{exactly } 0$ , then  $\bar{W}_0 = \infty$  and the energy (10.2) must be infinite for at least one of the vectors  $F_i^{(\alpha)}$ .

In the same way, we can consider concentrated forces of higher orders, described by the Dirac derivatives. For example, for the force

$$(10.5) \quad f_i(\mathbf{x}) = A_{im} \delta_{,m}^{(3)}(\mathbf{x}),$$

where  $A_{im}$  represents a certain matrix, the corresponding deformation energy is

$$(10.6) \quad W = -\frac{1}{2} G_{ij}(0) A_{im} A_{jm}.$$

Let  $\bar{W}_1$  denote the average energy corresponding to nine matrices  $A_{ij}$  such that

$$(10.7) \quad A_{im}^{(\alpha)} A_{im}^{(\beta)} = \delta^{\alpha\beta}.$$

Then we have

$$(10.8) \quad \bar{W}_1 = -\frac{1}{18} G_{ij, mn}(0) \sum_{\alpha=1}^9 A_{im}^{(\alpha)} A_{jn}^{(\alpha)} = -\frac{1}{18} \Delta G_{ii}(0)$$

and

$$(10.9) \quad \|L^{-1}\|_{-2} = 6 \bar{W}_0 + 18 \bar{W}_1.$$

Hence  $\bar{W}_1$  is finite, if and only if,

$$(10.10) \quad s(L^{-1}) \leq \text{almost } -2.$$

Analogous results are valid for concentrated forces of arbitrary orders.

Now, let us briefly discuss the interaction energy of two force centres having the form (10.5). Provided that the corresponding matrices  $A_{im}^I$  and  $A_{im}^{II}$  are symmetric, such force centres can be considered simple models of some point defects. Let  $r$  denote the relative position of the centres. Then, the interaction energy of the centres equals:

$$(10.11) \quad W^{\text{int}}(\mathbf{r}) = -\frac{1}{2} G_{ij, mn}(\mathbf{r}) (A_{im}^I A_{jn}^{II} + A_{im}^{II} A_{jn}^I).$$

If the defects are identical, then

$$(10.12) \quad W^{\text{int}}(\mathbf{r}) = -G_{ij, mn}(\mathbf{r}) A_{im} A_{jn}$$

and, in particular,

$$(10.13) \quad W^{\text{int}}(0) = 2W,$$

where  $W$  is given by the Eq. (10.6).

By applying theorem (6.3) to the operator  $L^{-1}$ , we conclude that the inequality (10.10) ensures finite values of the interaction energy (10.11) at all distances including  $r = 0$ . On the other hand, it follows from the inequality (8.1) that for the inequality (10.10) to be valid, the singular order of the operator  $L$  cannot be smaller than exactly 8.

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POLISH ACADEMY OF SCIENCES.

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