

The influence of viscosity on the stability of a relative motion of two media

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THE PAPER is devoted to an analysis of the influence of viscoelasticity of materials on the shape of the regions of instability of motion. The considerations concern linear models of bodies described by polynomial differential operators. As examples, instability regions were determined for the Voigt and Maxwell bodies. The critical velocities were determined on the basis of Mikhaylov and Nyquist stability criteria. It was proved that the stability of motion is influenced by both the flux of the dissipated energy and the ratios of the coefficients describing the viscoelastic properties of both media.

Pracę poświęcono analizie wpływu lepko-sprężystego charakteru ośrodków na ukształtowanie zakresów stateczności ruchu. Rozważania ogólne dotyczą liniowych modeli ciała, opisanych wielomianowymi operatorami różniczkowymi. W charakterze przykładów wyznaczono zakresy stateczności dla ośrodków o modelu Voigta i Maxwella. Prędkości krytyczne wyznaczono w oparciu o kryteria stateczności Michajłowa i Nyquista. Wykazano, że na stateczność układu ma wpływ zarówno strumień dysypowanej energii jak i ilorazy współczynników charakteryzujących lepkie własności obu ośrodków.

В работе дан анализ влияния вязкоупругого характера среды на пределы устойчивости движения. В общей части обсуждаются линейные модели тел, описываемые многочленными дифференциальными операторами. В качестве примеров найдены пределы устойчивости для сред, описываемых моделями Фойгта и Максвелла. Критические скорости определены на основе критерия устойчивости Михайлова и Никвиста. Показано, что на устойчивость системы влияют, как поток диссипируемой энергии, так и произведения коэффициентов, характеризующих вязкие свойства обеих сред.

1. Introduction

The stability of a relative motion of two media constitutes an important engineering problem and has extensively been investigated.

In one of the papers devoted to this problem [1] a relative motion of two elastic media was examined. It was proved that there exists a relative velocity above which the contact surface is deformed and takes the form of a travelling wave increasing in time.

The problem of the stability of motion constituted also the subject of a number of papers by S. KALISKI who investigated both mechanical systems and systems of coupled fields; one of the papers [2] dealt with two perfectly conducting media in a magnetic field perpendicular to the plane of motion.

In the above papers the influence of the viscosity of the media on the values of critical parameters was neglected. In view of the results of the papers [3–5] devoted mainly to the interaction between a moving system of oscillators and travelling waves in continuous media, exhibiting the important influence of the viscosity, it seems expedient to investigate

this influence on the relative motion of two media. In Sec. 2 of this paper we present the fundamental equations and the boundary conditions on the basis of viscoelasticity [6]. The third Sec. contains the solution of the equations while the fourth Sec. is devoted to an analysis of the influence of the viscosity on the generation and form of the regions of instability of the motion.

2. Equations of motion and boundary conditions

Consider two media: the first has density ρ and constants $a_\alpha^{(n)}, b_\alpha^{(n)}$ describing the viscoelastic material, while the second characterized by the constants $\rho^*, a_\alpha^{(n)*}, b_\alpha^{(n)*}$, is moving with respect to the first with a constant velocity V . If we associate with each body a coordinate system such that the motion occurs along the x_1 -axis and the solutions are independent of x_2 (Fig. 1), then the coordinates are connected by the relations

$$(2.1) \quad x_1 - x_1^* = Vt, \quad x_3 - x_3^* = 0.$$

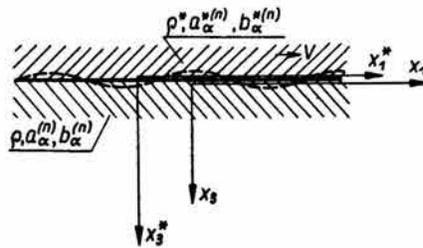


FIG. 1

The equations of motion of the viscoelastic medium will be written in the form

$$(2.2) \quad Q_1 P_2 \nabla^2 \ddot{\mathbf{u}} + 1/3(2Q_2 P_1 + Q_1 P_2) \text{grad div } \mathbf{u} - 2P_1 P_2 \rho \ddot{\mathbf{u}} = 0,$$

where P_α and Q_α denote the differential operators

$$(2.3) \quad P_\alpha = \sum_{m=0}^{N\alpha} a_\alpha^{(m)} \frac{\partial^m}{\partial t^m}, \quad a_\alpha^{(N\alpha)} \neq 0,$$

$$Q_\alpha = \sum_{n=0}^{M\alpha} b_\alpha^{(n)} \frac{\partial^n}{\partial t_\alpha^n}, \quad b_\alpha^{(M\alpha)} \neq 0.$$

If we neglect the friction on the contact surface of the two media and assume for the time being that the pressure normal to the plane of motion $p(x_1, t)$ is given, then the boundary conditions on the surface of the semi-space ($x_3 = 0$) take the form

$$(2.4) \quad \sigma_{33}|_{x_3=0} = -p(x_1, t), \quad P_1 \sigma_{33} = \left(KP_1 - \frac{Q_1}{3} \right) (u_{1,1} + u_{3,3}) + Q_1 u_{3,3},$$

$$\sigma_{13}|_{x_3=0} = 0, \quad P_1 \sigma_{13} = \frac{Q_1}{2} (u_{1,3} + u_{3,1}),$$

where $K = Q_2/3P_2 = \text{const}$ (assuming that our medium behaves in pure compression as an elastic medium).

Completing (2.4) by the condition of compatibility of the pressures and displacements of the two media on the contact surface in one of the coordinate systems, e.g.,

$$(2.5) \quad \begin{aligned} u_3(x_1, t) &= u_3^*(x_1, t), \\ p(x_1, t) &= p^*(x_1, t), \end{aligned}$$

we arrive at a system of equations and boundary conditions completely describing the problem.

We now proceed, therefore, to the solution of the above formulated problem and to a discussion of the characteristic equation.

3. Solution of the problem

We seek stationary solutions of Eqs. (2.2) in the form of the following travelling waves:

$$(3.1) \quad \begin{aligned} u_1 &= A_1(x_3) e^{ik(x_1 - vt)}, \\ u_3 &= A_3(x_3) e^{ik(x_1 - vt)}. \end{aligned}$$

Since the required solutions are periodic, as well as the pressure acting on the surface, we can make use of Alfrey's elastic-viscoelastic analogy and write Eqs. (2.2) in the form

$$(3.2) \quad \hat{\mu}(ikv) \nabla^2 \mathbf{w} + [\hat{\mu}(ikv) + \hat{\lambda}(ikv)] \text{grad div } \mathbf{w} - k^2 v^2 \mathbf{w} = 0,$$

whereas the boundary conditions are

$$(3.3) \quad \hat{\sigma}_{13} = \hat{\mu}(ikv) (w_{1,3} + w_{3,1}) = 0,$$

$$(3.4) \quad \hat{\sigma}_{33} = \hat{\lambda}(ikv) (w_{1,1} + w_{3,3}) + 2\hat{\mu}(ikv) w_{3,3} = -\hat{p}(x_1, kv),$$

where

$$\begin{aligned} \sigma_{sj}(x_1, t) &= \text{Re}[\hat{\sigma}_{sj}(x_1, kv) e^{ik(x_1 - vt)}], \\ u_j(x_1, t) &= \text{Re}[w_j(x_1, kv) e^{ik(x_1 - vt)}], \\ p(x_1, t) &= \text{Re}[\hat{p}(x_1, kv) e^{ik(x_1 - vt)}]. \end{aligned}$$

Substituting into (3.2) the solutions in the form (3.1), we arrive at a system of two ordinary differential equations with constant coefficients, the solutions of which have the form

$$(3.5) \quad A_1(x_3) = \sum_{s=1}^4 q_s e^{kr_s x_3},$$

$$(3.6) \quad A_3(x) = \sum_{s=1}^2 r_s q_s e^{kr_s x_3} + \sum_{n=3}^4 \frac{1}{r_n} q_n e^{kr_n x_3},$$

where

$$(3.7) \quad r_{1,2} = \pm \sqrt{1 - \frac{\rho v^2}{\hat{\mu}}}, \quad r_{3,4} = \pm \sqrt{1 - \frac{\rho v^2}{\hat{\lambda} + 2\hat{\mu}}}.$$

To satisfy the radiation condition, we neglect two terms in each of the solutions (3.5) and (3.6) which do not satisfy the inequality

$$(3.8) \quad \operatorname{Re}(kr_s) < 0, \quad s = 1, 2, 3, 4.$$

The roots r_s satisfying the condition (3.8) will be denoted by r_1 and r_3 . The boundary condition (3.3) is now employed to determine the relation between the constants C_1 and C_3 . Next, we make use of the condition (3.4) and the solutions (3.5) and (3.6) to derive relations between the pressure acting on the surface and the displacements of the medium, namely

$$(3.9) \quad \hat{w}_1 = \frac{i\hat{p}(x_1, kv)}{\hat{\mu}k} \frac{-(1+r_3^2)e^{kr_1x_3} + 2r_1r_3e^{kr_3x_3}}{(1+r_3^2)^2 - 4r_1r_3},$$

$$(3.10) \quad w_3 = \frac{\hat{p}(x_1, kv)}{\hat{\mu}k} \frac{-(1+r_3^2)r_1e^{kr_1x_3} + 2r_1e^{kr_3x_3}}{(1+r_3^2)^2 - 4r_1r_3}.$$

An analogous reasoning holds for the moving body described by ϱ^* , $\hat{\mu}^*$, $\hat{\lambda}^*$. The form of the derived relations in the coordinate system x_1^* , x_3^* is the same, while the indices of the roots undergo a change, since the condition (3.8) in the moving system is satisfied only for the roots r_2^* and r_4^* .

Thus, in the moving system we have

$$(3.11) \quad w_1^* = \frac{i\hat{p}(x_1^*, k^*v^*)}{\hat{\mu}^*k^*} \frac{-(1+r_4^{*2})e^{k^*r_2^*x_3^*} + 2r_2^*r_4^*e^{k^*r_4^*x_3^*}}{(1+r_4^{*2})^2 - 4r_2^*r_4^*},$$

$$(3.12) \quad w_3^* = \frac{\hat{p}^*(x_1^*, k^*v^*)}{\hat{\mu}^*k^*} \frac{-r_1^*(1+r_4^{*2})e^{k^*r_2^*x_3^*} + 2r_2^*e^{k^*r_4^*x_3^*}}{(1+r_4^{*2})^2 + 4r_2^*r_4^*}.$$

The condition of compatibility of the pressures and displacements on the boundary of the two media and the relations between the stationary and moving coordinate systems (2.1) yield the relations

$$(3.13) \quad k^* = k, \quad v^* = v - V.$$

The displacements on the contact surface of the media (i.e. for $x_3 = 0$) take the form

$$(3.14) \quad w_3|_{x_3=0} = \frac{\hat{p}(x_1, kv)}{\hat{\mu}k} \frac{r_1(1-r_3^2)}{(1+r_3^2)^2 - 4r_1r_3},$$

$$(3.15) \quad w_3^*|_{x_3^*=0} = \frac{\hat{p}^*(x_1^*, k^*v^*)}{\hat{\mu}^*k^*} \frac{r_2^*(1-r_4^{*2})}{(1+r_4^{*2})^2 - 4r_2^*r_4^*}.$$

The expression (3.14) can be written as follows:

$$w_3 = \frac{\hat{p}(x_1, kv)}{\mu_0 k \Phi(\beta, \gamma, v_0)},$$

where

$$\Phi(\beta, \gamma, v_0) = \frac{(2 - \beta v_0^2)^2 - 4\sqrt{1 - \beta v_0^2}\sqrt{1 - \beta\gamma v_0^2}}{\beta^2 v_0^2 \sqrt{1 - \beta\gamma v_0^2}}$$

and

$$\gamma = \frac{\hat{\mu}}{\hat{\lambda} + 2\hat{\mu}}, \quad \beta = \frac{\mu_0}{\hat{\mu}}, \quad v_0^2 = \frac{\rho v^2}{\mu_0}.$$

From a comparison of the relations (3.14) and (3.15) it follows [on the basis of (3.6)] that the forms of the functions differ only in sign.

To derive the characteristic equation of our problem, we make use of the condition (2.6), i.e., we equate the real values of the expressions (3.16) for the stationary and moving media. The compatibility of the amplitude of the wave takes the form

$$(3.18) \quad \frac{\operatorname{sgn}[\Phi(\beta, \gamma, v_0)]}{\mu_0 \sqrt{\{\operatorname{Re}[\Phi(\beta, \gamma, v_0)]\}^2 + \{\operatorname{Im}[\Phi(\beta, \gamma, v_0)]\}^2}} = \frac{\operatorname{sgn}[\Phi^*(\beta^*, \gamma^*, v_0^*)]}{\mu_0^* \sqrt{\{\operatorname{Re}[\Phi^*(\beta^*, \gamma^*, v_0^*)]\}^2 + \{\operatorname{Im}[\Phi^*(\beta^*, \gamma^*, v_0^*)]\}^2}}.$$

Similarly to [4], this equation has to be completed by the condition of compatibility of the phase displacements. Since the considered system is linear, the conditions of compatibility of the amplitude and the phase can be replaced by equating the real and imaginary parts of both characteristic functions, namely

$$(3.19) \quad \Phi(\beta, \gamma, v_0) = \Phi^*(\beta^*, \gamma^*, v_0^*).$$

In equating the amplitudes and the phases we make use of the following property of the system: the unstable motion is continuously generated from the stable motion (this can be proved on the basis of the Nyquist criterion [1]). Thus, in determining the critical states we may confine ourselves to the determination of the parameters characterising periodic solutions which neither decrease nor increase in time.

In the second case, we may directly apply to Eq. (3.19) the Mikhaylov stability criterion which makes it possible to verify whether for a selected value of V the system is stable.

Since both $\operatorname{Re} \Phi(\beta, \gamma, v_0)$ and $\operatorname{Im} \Phi(\beta, \gamma, v_0)$ are continuous functions of v_0 , the critical parameters of the motion determine real v_0, v_0^* constituting the solution of Eq. (3.19). Thus, this equation together with the second relation (3.13) constitute the complete system of characteristic equations of our problem.

In the next Section we shall determine the critical velocities of motion for two particular cases of the operators (2.3) describing the viscoelastic Voigt and Maxwell materials.

4. Critical parameters

Setting in the formulae for the differential operators P_α and Q_α , $N_1 = 0$, $M_1 = 0$, $N_2 = 0$, $M_2 = 0$, and substituting the appropriate values of $a_\alpha^{(0)}$ and $b_\alpha^{(n)}$ ($n = 0, 1$), we arrive at a Voigt model, for which the functions constituting the generalization of the Lamé constants to the viscoelastic medium, take the form

$$(4.1) \quad \hat{\mu} = \mu_0 + i\delta kv, \quad \hat{\lambda} = K - 2/3(\mu_0 + i\delta kv).$$

The characteristic function (3.17) containing the characteristic polynomial of the Rayleigh waves for the Voigt body takes the form

$$(4.2) \quad \Phi_V(\eta, \gamma_0, v_0) = -\frac{(2-v_0^2)^2 - 4\eta^2 v_0^2 + 4i\eta v_0(2-v_0^2)}{v_0^2 \sqrt{1-v_0^2(9\gamma_0 - 12\gamma_0^2)(1-16\eta^2 v_0^2)^{-1}}} - \frac{4(1-\eta^2 v_0^2 + 2i\eta v_0)}{v_0^2} \sqrt{1 - \frac{v_0^2(1-i\eta v_0)}{1+\eta^2 v_0^2}},$$

where

$$\eta = \frac{\delta k}{\sqrt{\rho\mu_0}}, \quad \gamma_0 = \frac{\mu_0}{\lambda_0 + 2\mu_0}.$$

A numerical analysis of the expression (4.2) implies that for a fixed pressure an increase of the coefficient η describing the viscosity is connected with an amplitude in the range of the velocities $|v|$ smaller than the Rayleigh wave velocity v_R .

The region of the phase plane for the case $\mu_0 = \mu_0^*$, $\lambda_0 = \lambda_0^*$, $\lambda_0 = \gamma_0^* = 0,3$ is presented in Fig. 2. The continuous part of the curve 1 is the same as in [1] for the elastic

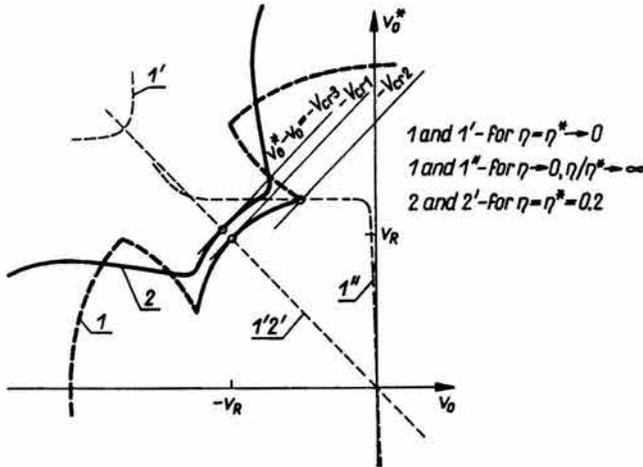


FIG. 2.

case and the critical velocity v_{cr1} determined in this paper is the same as the critical velocity for viscoelastic bodies when $\eta \rightarrow 0$ and $\eta/\eta^* = 1$.

In the case when the value of η is sufficiently small, the influence of the viscosity on the position of the curve 1 describing the real part of Eq. (3.19) is negligible. The configuration of the curves describing the imaginary part of Eq. (3.19) depends on the ratio η/η^* and for $\eta \gg \eta^*$ takes a position between the lines 1 and 1'. Thus, there exists a possibility of a significant change of the critical velocity of the relative motion of the media within the range $v_{cr1} \geq v_{cr} \geq v_{cr2}$, when the viscoelastic nature of the body is taken into account, even for small values of the coefficient η .

The case $v_{cr} > v_{cr1}$ occurs when the viscosity coefficients of both bodies increase and $\eta/\eta^* \sim 1$ (e.g., the case of v_{cr3}).

It is evident that an exchange of the coefficients η and η^* does not influence the critical velocity of motion, since the image on the phase plane v, v^* will be symmetric with respect to the straight line $v = -v^*$.

The Maxwell model is obtained by setting in the formulae (2.3) $N_1 = 1, M_1 = 1, N_2 = 0, M_2 = 0$ and introducing the appropriate values of the coefficients a_α and b_α [6]. The quantities $\hat{\lambda}$ and $\hat{\mu}$ are then the following:

$$(4.3) \quad \hat{\mu} = \mu_0 \frac{i\delta kv}{\mu_0 + i\delta kv}, \quad \hat{\lambda} = K - \frac{2i\mu_0 \delta kv}{3(\mu_0 + i\delta kv)},$$

while the characteristic function given by the formula (3.17), takes the form

$$(4.4) \quad \Phi_M(\eta, \gamma_0, v_0) = \left(- \left[\left[(2 - v_0^2)^2 - \frac{v_0^2}{\eta^2} \right] (1 - \eta^{-2} v_0^{-2}) - 4(2 - v_0^2) \eta^{-2} \right. \right. \\ \left. \left. + i \left\{ \left[(2 - v_0^2)^2 - \frac{v_0^2}{\eta^2} \right] \frac{2}{\eta v_0} + 2 \frac{v_0}{\eta} (2 - v_0^2) (1 - \eta^{-2} v_0^{-2}) \right\} \right] : \left[v_0^2 (1 + \eta^{-2} v_0^{-2}) \times \right. \right. \\ \left. \left. \times \sqrt{1 - v_0^2 \frac{9(1 + \eta^2 v_0^2) \gamma_0 - 12\gamma_0^2 + i[9(1 - \eta v_0) \gamma_0 - 12]}{(1 + 4\gamma_0)^2 + 9\eta^2 v_0^2}} \right] \right) \\ \left. - 4 \frac{\sqrt{1 - v_0^2 + i\eta^{-1} v_0 (1 - \eta^{-2} v_0^{-2} + 2i\eta^{-1} v_0^{-1})}}{v_0^2 (1 + \eta^{-2} v_0^{-2})} \right).$$

The results of a numerical analysis of the characteristic equations for the Maxwell body are presented in Fig. 3.

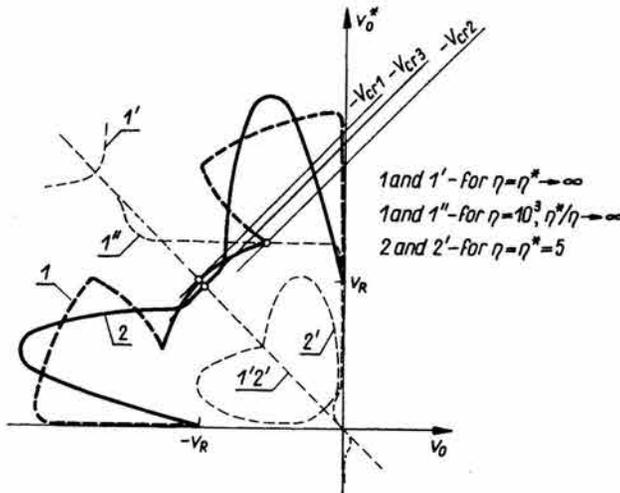


FIG. 3

The transition to elastic body is obtained by setting $\eta = \eta^* \rightarrow \infty$. For a finite value of the coefficient η , the curves constituting the image of the real part of Eq. (3.19) degenerate in a different manner than for the Voigt body. The unstable motion of the Maxwell bodies for $\eta = \eta^*$ and the assumed value of the coefficient γ_0 occurs for velocities smaller

than those determined in [1] for the case of elastic medium. The critical velocities of motion of Maxwell bodies depend also on the ratio η/η^* .

There is a new qualitatively different from others property in Fig. 3, namely the appearance of additional singular solutions with the parameters $v = 0$, $v^* = v_R$ and $v^* \neq 0$, $v = v_R$ occurring as $\eta \rightarrow \infty$, $\eta/\eta^* \rightarrow \infty$. The existence of these solutions suggests a possibility of a divergent loss of stability of one of the media and oscillatory of the other one. Since the equations of motion (3.2) were derived under the assumption of a periodic nature of the solutions (in time), a doubt arises whether they hold for a divergent solution. However, the fact that the curves representing the real and imaginary parts of Eq. (3.19) can be at most tangent, which in accordance with Mikhaylov stability criterion leads to a possibility of a non-increasing periodic solution; hence, $V = v$ is not a critical velocity.

It follows from calculations that the maximum critical velocity in the system for $\eta = \eta^* \rightarrow \infty$ is $2v_R$.

Consider one more limiting case. Suppose that one of the bodies is almost elastic $\eta_0 \rightarrow 0$ or $\eta_M \rightarrow \infty$, while the other one perfectly elastic, e.g., $\eta_0 = 0$. The critical velocity in this case will differ by a finite value from the critical velocity of a relative motion of two perfectly elastic bodies. This seemingly contradictory fact can readily be explained by an analysis of the decrements of the waves which in the range of velocities $v_{cr1} \geq V \geq v_{cr2}$ depend also on the dissipated by viscosity energy of the system. This phenomenon was also investigated in [3] where, among others, the authors considered a damped plate in a potential gas flow. It was proved in this paper by an analysis of the behaviour of non-stationary solutions, that the fact that an infinitesimal damping of the plate leads to a finite difference in the critical velocities, is due to the infinitely large time of the process.

5. Concluding remarks

It follows from our considerations that the influence of viscosity on the critical velocity of motion (relative motion) of two media may, in certain cases be significant.

The main conclusions are the following.

1. The relatively large range of critical velocities of relative motion depends on the relation between the viscosity coefficients of the moving media.
2. For sufficiently small viscosity coefficients for Voigt materials, there exists a possibility of a loss of stability for velocities smaller than the critical velocity of the relative motion of elastic media ($\eta = 0$).
3. For $\eta/\eta^* \sim 1$, the values of the critical velocities of the relative motion increase as η increases, both in the case of Voigt and Maxwell bodies ($\gamma_0 = 0.3$).

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Received October 18, 1971
