

Inelastic buckling of plate

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THIS PAPER concerns the inelastic stability of a thin plate under in-plane loading. In Love–Kirchhoff's approximation, using Hencky's relations and Von Mises criterion, we can obtain the particular stress distribution across the plate thickness for elastic and elastic-plastic prestress fields. The principle of virtual work is used to study the equilibrium of the bifurcated solution. This leads to the energy relations where explicit dependence between the stability equation coefficients and the solution is carried out. We obtain, in the equilibrium equation, two non-quadratic additional terms, which are neglected in the classical equation. Some applications are made using Ramberg–Osgood's formula to show the importance of the additional terms.

Notations

Superscripts

- T total value,
- 0 the value calculated at bifurcation point,
- p plastic value.

Subscript

- $\alpha, \beta = 1, 2$ for a tensor component,
- $i, j = 1, 2, 3$
- ε_{ij}, e_{ij} strain components and deviatoric ones,
- σ_{ij}, S_{ij} stress components and deviatoric ones,
- U_1 displacement components,
- z coordinate across the plate thickness,
- t charge parameter,
- E, G, φ, ν parameters of constitutive relations,
- $\bar{\sigma}, \bar{\varepsilon}$ effective stress and strain,
- $\bar{\sigma}^e = f(\bar{\varepsilon})$ strain-stress curve.

1. Introduction

INELASTIC stability of plates and shells has been intensively studied for the last 40 years (see, for example, BUSHNELL, HUTCHINSON, RERKSHANANDANA, GALLAGHER, and their bibliography [1, 4]). The stability can be investigated by a variational method like the virtual work principle. We can establish that the eigenvalue problem has the same form as that in the elastic case, but with coefficients depending on the bifurcated solution which complicates the resolution. This paper concerns the bifurcation load for a thin plate under in-plane loading. The objective is to show explicitly this dependence between the stability equation coefficients and the bifurcated solution in the case of elastic-plastic behaviour. We shall take into account the particular stress distribution across the plate thickness and obtain two additional non-quadratic terms, as compared with the classical equation.

To obtain the explicit energy relations it is necessary to integrate the incremental constitutive relations, and here Hencky's relations will be used. This approximation is

justified by the well known fact that the deformation theory of plasticity gives bifurcated loads which are in reasonable agreement with the experimental results [2, 5]. Under this assumption, equilibrium relations are obtained with different expressions depending on whether the pre-stress field is elastic or elastic-plastic.

Applications are made to the case of uniform in-plane loading with the Ramberg-Osgood's material stress-strain curve. The result shows the importance of the additional terms.

2. Kinematics

2.1. Assumptions concerning buckling of thin plates

Classical assumptions are adopted [3, 5, 6]. We assume that the plate thickness is very small as compared with other dimensions. We study the beginning of the deflection. During this phase, strains and displacements are considered to be small and are obtained by means of Love-Kirchhoff's approximation. Under these assumptions, in a general system of coordinates, total deformations are [7]:

$$(2.1) \quad \varepsilon_{\alpha\beta}^T = \varepsilon_{\alpha\beta}^0 + (-zU_{3,\alpha\beta}) + 1/2(U_{\alpha,\beta} + U_{\beta,\alpha} + U_{3,\alpha} \cdot U_{3,\beta}).$$

Here with $x_3 \equiv z$:

- 1st term = pre-buckling deformation,
- 2nd term = pure bending deformation,
- 3rd term = deformation of middle surface.

At the beginning of the deflection process, stretching of the middle surface is supposed to be negligible as compared with other strains,

$$(2.2) \quad U_{\alpha,\beta} + U_{\beta,\alpha} = -U_{3,\alpha} \cdot U_{3,\beta} \quad (\alpha, \beta = 1, 2).$$

2.2. Deformation

At the beginning of buckling, parameters which lead to a certain bifurcated geometric form are nearly constant; hence, the general form of the deflection surface is assumed to be constant, only its intensity varies.

$$(2.3) \quad U_3(t, x_\alpha) = t \cdot U_3(x_\alpha).$$

Loading parameter t varies from 0 to 1.

3. Material relations

Classical elastic-plastic behaviour is modelled by the Prandtl-Reuss relations. The objective of this study is to obtain explicit energy relations according to the particular stress distribution across the thickness. So we need to integrate the stress-strain relations. Since the calculations based on the deformation theories of plasticity lead to a reasonably good agreement with the experiments [2, 5], proportionality between the deviatoric components of stress and strain is assumed.

The Prandtl-Reuss equations can be integrated and lead to HENCKY's relations [8],

$$(3.1) \quad e_{ij} = \left(\varphi + \frac{1}{2G} \right) S_{ij} \quad \text{and} \quad \varepsilon_{ii} = \frac{(1-2\nu)}{E} \sigma_{ii}.$$

The Von Mises yield criterion is used [9],

$$(3.2) \quad \bar{\sigma} = \sqrt{\frac{3}{2}} (S_{ij} \cdot S_{ij})^{1/2} \leq \bar{\sigma}^e$$

and isotropic hardening rule is assumed,

$$(3.3) \quad \bar{\sigma}^e = f(\bar{\varepsilon}^p).$$

Since the plate is loaded in its plane (no out-of-plane shear forces), plane stress conditions are assumed, and

$$\sigma_{13} = 0.$$

NB: In general, if we take an incremental behaviour law like

$$\overset{\circ}{S} = k \overset{\circ}{e},$$

the following expressions have the same form but for a step of time.

4. Stress distribution

Two different cases are obtained, depending upon whether the pre-stress field is elastic or elastic-plastic.

4.1. Elastic pre-stress

With precedent assumptions we obtain the transversal deformation ε_{33} in the plate.

$$\varepsilon_{33} = \frac{(1-2\nu)(\varepsilon_{11} + \varepsilon_{22})}{(3(1-\nu) + \varphi E)}.$$

Simple calculation yields the deviatoric and later the stress tensor components,

$$(4.1) \quad \sigma_{11} = \frac{E(-z) \cdot t}{\det} [U_{3,11} \cdot (3 + 2\varphi E) + U_{3,22}(3\nu + \varphi E)] \\ + \overset{-1}{\det} [\sigma_{11}^0(3(1-\nu^2) + (2-\nu)\varphi E) + \sigma_{22}^0(1-2\nu)\varphi E],$$

$$\sigma_{12} = ((1+\nu) + \varphi E)^{-1} [E \cdot U_{3,12}(-z)t + \sigma_{12}^0(1+\nu)],$$

with

$$\det = (1+\nu + \varphi E) \cdot (3(1-\nu) + \varphi E).$$

4.2. Elastic-plastic pre-stress

At the bifurcation point, an initial plastic deformation appears.

For example, in direction 1

$$(4.2) \quad \sigma_{11}^0 = \frac{1}{\varphi^0} (2\varepsilon_{11}^{op} + \varepsilon_{22}^{op})$$

and during the deflection growth

$$\sigma_{11}^T = \frac{1}{\varphi}(2\varepsilon_{11}^p + \varepsilon_{22}^p) + \frac{\varphi^0}{\varphi}\sigma_{11}^0$$

the expressions for stresses differ slightly from those in (4.1).

$$\begin{aligned} \sigma_{11}^T &= \frac{E(-z)t}{\det} [U_{3,11}(3 + 2\varphi E) + U_{3,22}(3\nu + \varphi E)] \\ &\quad + \frac{-1}{\det} [\sigma_{11}^0(3(1 - \nu^2) + (2 - \nu)\varphi E) + \sigma_{22}^0(1 - 2\nu)\varphi E] \\ &\quad \cdot \left[1 - \frac{\varphi^0}{\varphi} \right] + \sigma_{11}^0 \cdot \frac{\varphi^0}{\varphi}, \\ (4.3) \quad \sigma_{12}^T &= \frac{E(-z)t}{(1 + \nu + \varphi E)} U_{3,12} + \sigma_{12}^0 \left[\frac{(1 - \nu)}{(1 + \nu + \varphi E)} \left(1 - \frac{\varphi^0}{\varphi} \right) + \frac{\varphi^0}{\varphi} \right]. \end{aligned}$$

REMARKS

In the second term of Eqs. (4.1) and (4.3) the first parts are due to bending, and the second parts due to the pre-stress field.

For example, a simplified expression of stress is

$$(4.4) \quad \sigma_{11} = (-z) \cdot t \cdot (A \cdot U_{3,11} + B \cdot U_{3,22}) + C \cdot \sigma_{11}^0 + D \cdot \sigma_{22}^0$$

with $(A, B, C, D) =$ function of (φ, E) .

We note that coefficients A, B, C, D are dependent or „coupled” by the material behaviour, and because of elastic-plasticity, they are not symmetric with respect to the middle surface in x_3 . Compared with elastic relations, the new aspect is a coefficient C different from 1 and a coefficient D different from 0. In elasticity $\varphi = 0 \rightarrow C = 1$.

4.3. Governing equations

We use the principle of virtual work to study the equilibrium of the bifurcated solution and the exact solution is approached by a linear combination of kinematic admissible functions,

$$\int_v \sigma_{\alpha\beta} \cdot \delta\varepsilon_{\alpha\beta} dx - \int_s T_\alpha \cdot \delta U_\alpha ds = 0.$$

Using the assumption that there is no stretching of the middle surface for strain and the local equilibrium equation for the stress, the virtual work of applied loading can be obtained only by means of the bifurcated shape U_3 , which then remains the only unknown.

$$\begin{aligned} \int_s T_\alpha \delta U_\alpha ds &= - \int_s \sigma_{\alpha\beta} \cdot n_\beta \cdot \delta U_\alpha ds \\ &= - \int_v \sigma_{\alpha\beta,\alpha} \cdot \delta U_\alpha dv - \int_v \sigma_{\alpha\beta} \cdot \delta U_{\alpha,\beta} dv = + \int_v \sigma_{\alpha\beta} \cdot U_{3,\alpha} \cdot \delta U_{3,\beta} \cdot t dv. \end{aligned}$$

This leads to the total virtual work

$$(4.5) \quad \int_v \sigma_{\alpha\beta} (\delta U_{3,\alpha\beta} - t \cdot U_{3,\alpha} \cdot \delta U_{3,\beta}) dv = 0.$$

With assumption (2.3) and global discretization,

$$(4.6) \quad U_3(t) = f_i \cdot t \cdot U_{i3},$$

we obtain a virtual displacement expression which is a variation of (4.6), so that $\delta U_3(t)$ and $\delta U_{\alpha,\beta}$ are expressed by the functions:

$$(4.7) \quad U_{i3} \quad \text{and} \quad 1/2 f_i \cdot U_{i3} \cdot U_{j3}.$$

Finally, f_i parameters are the unknowns of the problem.

We introduce stress expression (4.1) or (4.2) in (4.5), and with the help of (4.6) we obtain the equation system

$$(4.8) \quad \sum_j \delta W_{ij} = 0, \quad i, j = 1, \dots, n$$

In this system there appear two classical terms δW_{i1} and δW_{i3} , quadratic in U_3 , and two additional non-quadratic terms δW_{i2} and δW_{i4} .

The quadratic term of bending

$$\delta W_{i1} = \int [a_1(U_{i3,11} \cdot U_{j3,11} + U_{i3,22} \cdot U_{j3,22}) + b_1(U_{i3,11} \cdot U_{j3,22} + U_{j3,11} \cdot U_{i3,22}) + 2c_1 \cdot U_{i3,12} \cdot U_{j3,12}] \cdot f_j \cdot ds,$$

with

$$\begin{aligned} a_1 &= \int E \cdot z^2 \cdot t \cdot (3 + 2\varphi E) \det^{-1} dz dt, \\ b_1 &= \int E \cdot z^2 \cdot t(3\nu + \varphi E) \det^{-1} dz dt, \\ c_1 &= \int E \cdot z^2 \cdot t(1 + \nu + \varphi E)^{-1} dz dt. \end{aligned}$$

The non-quadratic term of bending:

$$\delta W_{i2} = \int [a_2(U_{i3,11} \cdot \sigma_{11}^0 + U_{i3,22} \cdot \sigma_{22}^0) + b_2(U_{i3,11} \cdot \sigma_{22}^0 + U_{i3,22} \cdot \sigma_{11}^0) + 2c_2 \cdot U_{i3,12} \cdot \sigma_{12}^0] ds,$$

with, for the elastic pre-stress,

$$\begin{aligned} a_2 &= \int (-z)(3(1 - \nu^2) + (2 - \nu)\varphi E) \det^{-1} dz dt, \\ b_2 &= \int (-z)((1 - 2\nu)\varphi E) \det^{-1} dz dt, \\ c_2 &= \int (1 + \nu)(-z)(1 + \nu + \varphi E)^{-1} dz dt \end{aligned}$$

and for the elastic-plastic pre-stress

$$\begin{aligned} a_2 &= \int (-z) \left[(3(1 - \nu^2) + (2 - \nu)\varphi E) \det^{-1} \left(1 - \frac{\varphi_0}{\varphi} \right) + \frac{\varphi_0}{\varphi} \right] dt dz, \\ b_2 &= \int (-z)(1 - 2\nu)\varphi E \cdot \det^{-1} \left(1 - \frac{\varphi_0}{\varphi} \right) dt dz, \\ c_2 &= \int (-z) \left((1 + \nu + \varphi E)^{-1} \left(1 - \frac{\varphi_0}{\varphi} \right) + \frac{\varphi_0}{\varphi} \right) dt dz. \end{aligned}$$

The quadratic term of the applied loads

$$\begin{aligned} \delta W_{i3} &= \int [a_3(\sigma_{11}^0 \cdot U_{i3,1} \cdot U_{j3,2} + \sigma_{22}^0 \cdot U_{i3,1} \cdot U_{j3,2}) \\ &\quad + b_3(\sigma_{22}^0 \cdot U_{i3,2} \cdot U_{j3,2} + \sigma_{11}^0 \cdot U_{i3,1} \cdot U_{j3,1}) \\ &\quad + c_3 \cdot \sigma_{12}^0 (U_{i3,1} \cdot U_{j3,2} + U_{i3,2} \cdot U_{j3,1})] f_j ds \end{aligned}$$

with, for the elastic pre-stress,

$$\begin{aligned} a_3 &= \int (3(1 - \nu^2) + (2 - \nu)\varphi E) \cdot t \cdot \det^{-1} dz dt, \\ b_3 &= \int (1 - 2\nu) \cdot \varphi \cdot E \cdot t \cdot \det^{-1} dz dt, \\ c_3 &= \int (1 + \nu) \cdot t(1 + \nu + \varphi E)^{-1} dt, \end{aligned}$$

and for the elastic-plastic pre-stress

$$\begin{aligned} a_3 &= \int \left((3(1 - \nu^2) + (2 - \nu)) \cdot \varphi \cdot E \cdot \det^{-1} \left(1 - \frac{\varphi_0}{\varphi} \right) + \frac{\varphi_0}{\varphi} \right) t \cdot dz dt, \\ b_3 &= \int (1 - 2\nu) \cdot \varphi \cdot E \cdot \det^{-1} \left(1 - \frac{\varphi_0}{\varphi} \right) t \cdot dz dt, \\ c_3 &= \int \left((1 + \nu)(1 + \nu + \varphi E)^{-1} \left(1 - \frac{\varphi_0}{\varphi} \right) + \frac{\varphi_0}{\varphi} \right) t \cdot dz dt. \end{aligned}$$

A non-quadratic term of applied loads:

$$\begin{aligned} \delta W_{i4} &= \int [a_4(U_{k3,11} \cdot U_{j3,1} \cdot U_{i3,1} + U_{k3,22} \cdot U_{j3,2} \cdot U_{i3,2}) \\ &\quad + b_4(U_{k3,22} \cdot U_{j3,1} \cdot U_{i3,1} + U_{k3,11} \cdot U_{j3,2} \cdot U_{i3,2}) \\ &\quad + c_4 \cdot U_{k3,12}(U_{j3,1} \cdot U_{i3,2} + U_{j3,2} \cdot U_{i3,1})] f_j f_k ds, \end{aligned}$$

with, for the elastic pre-stress:

$$\begin{aligned} a_4 &= \int E(-z)t^2(3 + 2\varphi E) \det^{-1} dz dt, \\ b_4 &= \int E(-z)t^2(3\nu + \varphi E) \det^{-1} dz dt, \\ c_4 &= \int E(-z)t^2(1 + \nu + \varphi E)^{-1} dz dt, \end{aligned}$$

and for the elastic-plastic pre-stress

$$\begin{aligned} a_4 &= \int E(-z)t^2(3 + 2\varphi E) \det^{-1} \left(1 - \frac{\varphi_0}{\varphi} \right) dz dt, \\ b_4 &= \int E(-z)t^2(3\nu + \varphi E) \det^{-1} \left(1 - \frac{\varphi_0}{\varphi} \right) dz dt, \\ c_4 &= \int E(-z)t^2(1 + \nu + \varphi E)^{-1} \left(1 - \frac{\varphi_0}{\varphi} \right) dz dt. \end{aligned}$$

NB: φ is a function of z and t .

SOME REMARKS

These expressions can also be used with the virtual power principle, but in this case integration over t disappears.

In the classical equation non-quadratic terms don't appear, because the C and D terms in the stress expression (4.4) are always set respectively at 1 and 0. Coupling between A , B , C , D terms of (4.4) disappears.

In the case of elasticity or constant plasticity through the plate thickness, for example with a tangent modulus, non-quadratic terms also vanish.

The plane configuration can lose its uniqueness but not its stability, because of non-quadratic terms.

Additional terms are linear and cubic in f_i . We can expect that the absolute value of the linear term must be greater than the cubic one, which is confirmed by numerical results.

5. Equation of yield surface, determination of plastic part

When pre-stress is in elasticity, evolution is governed by $\bar{\sigma} < \bar{\sigma}^e \Rightarrow$ elasticity; $\bar{\sigma} > \bar{\sigma}^e \Rightarrow$ elasto-plasticity, with $\bar{\sigma}$ denoting the effective stress.

Using stress distribution (4.1), we obtain an equation in (z, t)

$$Az^2t^2 - Bzt + C = 0,$$

with

$$C = (\bar{\sigma}^0)^2 - (\bar{\sigma}_0^e)^2,$$

$$B = \frac{E}{(1 - \nu^2)} [\sigma_{11}^0((2 - \nu)U_{3,11} + (2\nu - 1)U_{3,22}) + \sigma_{22}^0((2 - \nu)U_{3,22} + (2\nu - 1)U_{3,11}) + 6\sigma_{12}^0(1 - \nu)U_{3,12}],$$

$$A = \frac{E^2}{(1 - \nu^2)^2} [(U_{3,11}^2 + U_{3,22}^2)(1 + \nu^2 - \nu) + U_{3,11} \cdot U_{3,22}(-\nu^2 + 4\nu - 1) + 3U_{3,12}^2(1 - \nu)^2].$$

NB: Term A is always positive for the reason that it corresponds to $\frac{(\bar{\sigma}^T)^2}{z^2t^2}$ when the pre-stress field is null.

Term C is always negative because the pre-stress point is elastic

$$(\bar{\sigma}^0)^2 < (\bar{\sigma}^e)^2.$$

The solution of the equation gives limits $z_\alpha(t)$ where plastic deformation occurs. These limits can be out of plane and are generally not symmetric with respect to the middle-plane in x_3 .

When the pre-stress point is in elasto-plasticity, evolution is governed by

$$d\bar{\sigma} > 0 \Rightarrow \text{elasto-plasticity}, \quad d\bar{\sigma} < 0 \Rightarrow \text{elasticity}.$$

We suppose that the strain-stress curve ($\bar{\sigma}^e = f(\bar{\varepsilon})$) always increases,

$$\frac{d\bar{\sigma}^e}{d\bar{\varepsilon}} > 0.$$

So, variations of effective stress and effective strain are equivalent. The development of $d\bar{\varepsilon}$ gives: $z^2 t A + (-z) B = 0$, with

$$A = (U_{3,11} + U_{3,22})^2 (2F + 1) + U_{3,11}^2 + U_{3,22}^2 + 2U_{3,12}^2,$$

$$B = (\varepsilon_{11}^0 + \varepsilon_{22}^0) (2F + 1) \cdot (U_{3,11} + U_{3,22}) + U_{3,11} \cdot \varepsilon_{11}^0 + U_{3,22} \cdot \varepsilon_{22}^0 + 2\varepsilon_{12}^0 \cdot U_{3,12},$$

and

$$F = \frac{3(1 - 2\nu)(2 - \nu + \varphi_0 \cdot E)}{(3(1 - \nu) + \varphi_0 E)^2}.$$

NB: In this equation, we don't take into account the variation of elastic voluminal deformation during the growth of the bifurcated solution. Voluminal deformation is then calculated at the bifurcation point. The solution of the equation gives limits $z_\alpha(t)$ with the same remarks as for the elastic pre-stress.

6. Numerical solution technique

Non-quadratic terms in the equilibrium equation are set to perturb the principal quadratic one. In reduced form, the bifurcation equation is

$$A_{ij} \cdot f_j + \lambda B_1 + \lambda C_{ij} \cdot f_j + D_{ijk} \cdot f_j \cdot f_k = 0$$

which is transformed to

$$\underbrace{\left[(A_{ij} + D_{ijk} \cdot f_k) + \lambda \left(C_{ij} + \frac{\delta_i^j B_i}{f_i} \right) \right]}_I f_j = 0.$$

Nontrivial solution implies

$$\det(I) = 0.$$

We use the iterative resolution

$$\det(\bar{A}_{ij}(\lambda^{(n)}, f_j^{(n)}) - \Delta \lambda^{(n+1)} \bar{C}_{ij}(\lambda^{(n)}, f_j^{(n)})) = 0,$$

with

$$\bar{A}_{ij}(\lambda^{(n)}, f_j^{(n)}) = A_{ij}(\lambda^{(n)}, f_j^{(n)}) + D_{ijk}(\lambda^{(n)}, f_j^{(n)}) \cdot f_k^{(n)},$$

$$\bar{C}_{ij}(\lambda^{(n)}, f_j^{(n)}) = C_{ij}(\lambda^{(n)}, f_j^{(n)}) + \delta_i^j \cdot \frac{B_i(\lambda^{(n)}) \cdot f_j^{(n)}}{f_i^{(n)}},$$

and

$$\lambda^{(n+1)} = \Delta \lambda^{(n+1)} \cdot \lambda^{(n)}.$$

In fact, in order to have a simple solution, just quadratic terms are used for the first iterations to localize the solution with nearly 10% of error. Then, non-quadratic terms are introduced to obtain the correct solution.

7. Applications

To show the influence of non-quadratic terms, rectangular plates with uniform in-plane loading are considered. All four edges are simply supported against out-of-plane displacements (Fig. 1). Initial deplanation with amplitude δ_0 can be simulated.

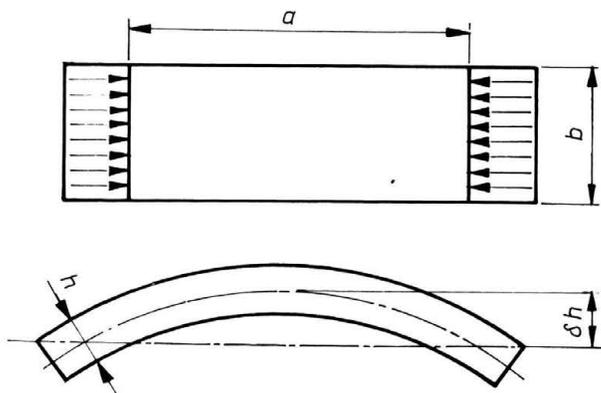


FIG. 1.

To have a different stress-strain curve, we adopt the RAMBERG-OSGOOD's formula [10]; σ , ε — uniaxial stress and strain, E — Young's modulus, α , n , σ_α — factors,

$$\varepsilon = \frac{\sigma}{E} + \frac{\alpha \sigma_\alpha}{E} \left(\frac{\sigma}{\sigma_\alpha} \right)^n \quad \text{Ramberg-Osgood formula.}$$

Discretization is made with only 3 functions,

$$U_3 = \sum_{i=1}^3 f_i \cdot U_{i3}.$$

Figure 2 shows the influence of initial deplanation compared with the incremental finite element result, in which it is necessary to initiate the bifurcated shape. Despite all approximations (simple behaviour law, no stretching of the middle surface), the results comply with the reference. We note that the influence of additional terms increases with the value of δ_0 .

The maximum effect of additional terms is nearly 3.6% and is much less important than the initial deplanation, here of a factor of 4.

Figure 3 shows a comparison with Shrivastava's results. The maximum difference between the two curves is 2%. The influence of additional terms increases when critical stress approaches elastic limits (≈ 400 MPa). Let us note the rapid variation of critical stress related to the ratio b/h .

To see the curvature effect of the stress-strain curve, we study two different cases, $n = 5$ and $n = 20$ in Ramberg-Osgood's formula. Figure 4 shows the difference between

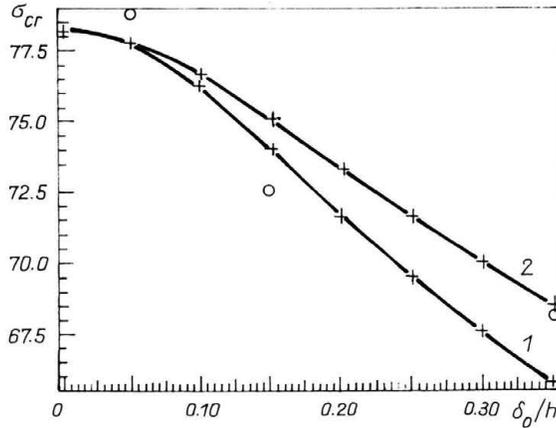


FIG. 2. The influence of initial deflection.

$$\varepsilon = \frac{\sigma}{70000} + 0.001 \left(\frac{\sigma}{100} \right)^5,$$

$$a/b = 0.6, b = 50.83 \text{ mm},$$

$$\nu = 0.33, h = 1 \text{ mm},$$

×1: with non-quadratic terms, ×2: without non-quadratic terms, □ 3: ref. G. H. LITTLE [11] p. 42.

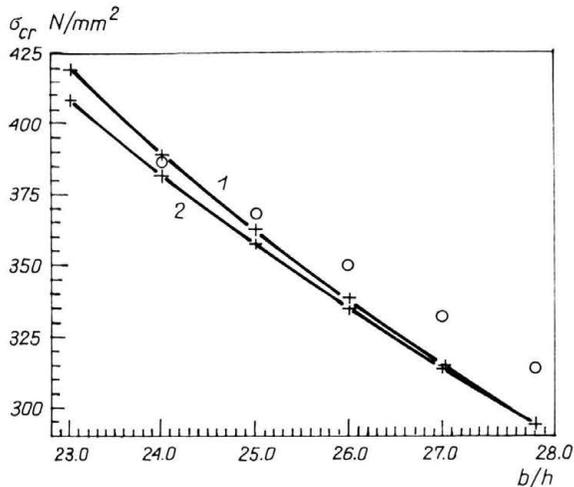


FIG. 3. Critical stress as a function of b/h for square plates simply supported on all sides.

$$\nu = 0.32, h = 1 \text{ mm},$$

$$\varepsilon = \frac{\sigma}{69015} + 0.001 \left(\frac{\sigma}{396.03} \right)^{20}.$$

1: with non-quadratic terms, 2: without non-quadratic terms, □: ref. S. C. SHRIVASTAVA [5]

the two curves and we note that the coefficient n decreases as the nonlinear part of the curve increases. For these two stress-strain curves, we indicate in Fig. 5 additional energy evolutions in the loading level.

The loading level is measured by the ratio between the current load and the critical load. Additional energies are measured by the ratio between their value and the value of the principal energy.

We notice that at the bifurcation point, the maximum difference is 5% and the differ-

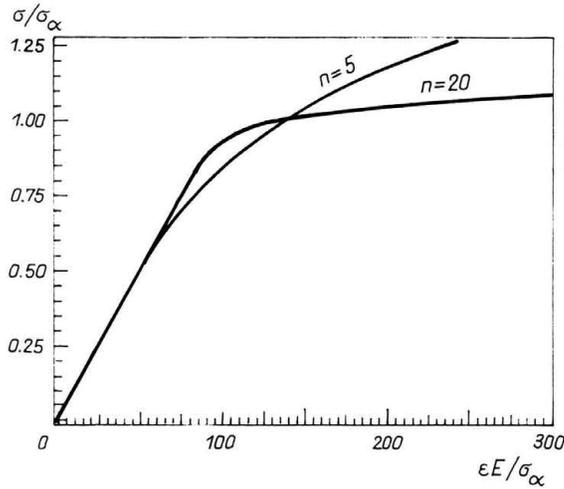


FIG. 4. Variation of Ramberg-Osgood's factor n .

$$\epsilon = \frac{\sigma}{E} + \frac{\alpha \sigma_\alpha}{E} \left(\frac{\sigma}{\sigma_\alpha} \right)^n,$$

$$E = 70000 \text{ N/m}^2, \alpha = 0.7,$$

$$\sigma_\alpha = 100 \text{ N/m}^2.$$

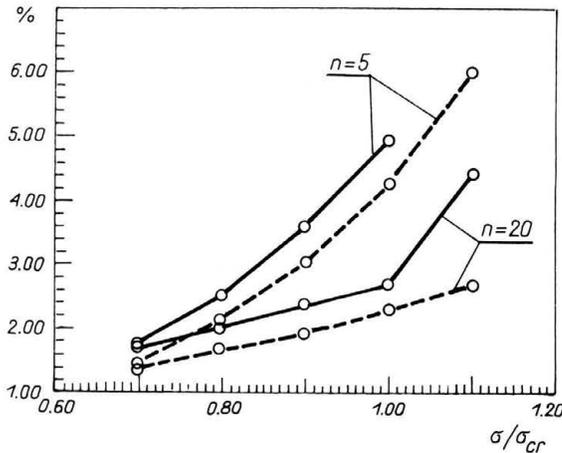


FIG. 5. Influence of Ramberg-Osgood's factor n on the value of non-quadratic terms

$$\left(\frac{\text{Resulting work for internal non-quadratic term}}{\text{total internal work}} \right) \times 100$$

$$\left(\frac{\text{Resulting work for external non-quadratic term}}{\text{total internal work}} \right) \times 100$$

Square plate simply supported, $b/h = 24$, $h = 1 \text{ mm}$.

ence between the two energies, quadratic and non-quadratic, either internal or external, is nearly constant. So, if additional terms are neglected, we make a systematic error almost constantly.

For the critical loading, the effect of additional terms is twice as great for $n = 5$ than for $n = 20$, which points out the influence of n .

8. Conclusion

In the classical equation of stability we have shown, for simple stress-strain relations, the explicit expression of residual membrane-bending terms which act in a non-quadratic manner. The influence of these terms is small but depends on the curvature of the stress-strain curve. We note that the critical stress predicted by all terms is always greater than the one predicted by the principal quadratic terms.

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