

A generalization of Faxén's theorems to include initial conditions

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IN THE PRESENT paper the time-dependent Faxén's theorems for a rigid sphere immersed in a viscous, incompressible fluid are derived. For this aim the Green's function, depending explicitly on the time variable, is applied. The Faxén's relations are obtained for the force and the torque exerted on the sphere in a time-dependent Stokes flow. The effect of the initial condition is taken into account.

W pracy tej wyprowadzono prawa Faxéna dla sztywnej kuli, umieszczonej w cieczy lepkiej, nieściśliwej, w warunkach niestacjonarnego przepływu cieczy (przepływ Stokesa) oraz niestacjonarnego ruchu kuli. Celem wyprowadzenia praw Faxéna zastosowano funkcję Greena, która zależy w sposób jawny od czasu. Prawa te dotyczą siły oraz momentu siły, działających na kulę ze strony niestacjonarnego przepływu cieczy. Uwzględniono wpływ, jaki wywiera na te wielkości początkowy rozkład prędkości w cieczy.

В этой работе выведены законы Факсена для жесткого шара, помещенного в вязкой, несжимаемой жидкости, в условиях нестационарного течения жидкости (течение Стокса), а также нестационарного движения шара. С целью вывода законов Факсена применена функция Грина, зависящая явным образом от времени. Эти законы касаются силы и момента силы, действующих на шар со стороны нестационарного течения жидкости. Учтено влияние, какое оказывает на эти величины начальное распределение скорости в жидкости.

1. Introduction

IN THE PAPER an unsteady motion of a single sphere immersed in an incompressible viscous fluid will be considered. In particular, in the case of time-dependent Stokes flows, the force and the torque exerted on the sphere will be calculated, taking into account the effect of initial conditions. The force and torque will be expressed in a closed form, namely, in the form of Faxén's relations. An understanding of these unsteady motions is much poorer than the understanding of such problems under steady conditions. A variety of steady problems has been recently presented in a review by LEAL [1].

Differences between steady and unsteady Stokes flows appear, for example, as a different asymptotic behaviour of fundamental singular solutions, for finite time, in the limit $|\mathbf{r}| \rightarrow \infty$. In the case of steady problems the solutions are $\mathbf{v}(\mathbf{r}) = O(|\mathbf{r}|^{-1})$, $p(\mathbf{r}) = O(|\mathbf{r}|^{-2})$, whereas for unsteady problems $\mathbf{v}(\mathbf{r}, t) = O(|\mathbf{r}|^{-3})$, $p(\mathbf{r}, t) = O(|\mathbf{r}|^{-2})$, for $t < \infty$. The other point is that Stokes flows, in which the influence of local acceleration can be omitted, possess the property of time reversibility; the velocity distribution for the backwards (time-reversed) flow is the reverse velocity distribution in the forward flow, whereas the pressure gradient in the forward flow is the negative of that in the backwards flow [2]. On the other hand, the Stokes flows, in which the local acceleration effects become de-

cisive, have a directional nature. Further differences between Stokes flows described by steady and unsteady equations of motion can be illustrated considering, for instance, the drag force on a rigid sphere. For a rigid sphere of radius a , moving slowly with the velocity \mathbf{U} in a viscous fluid, being at rest at infinity, the steady Stokes drag is $\mathbf{F} = -6\pi a\mu\mathbf{U}$. Under unsteady conditions the relation of the force to the sphere velocity is given by a linear operator called, after Boussinesq, the B operator:

$$(1.1) \quad B = -6\pi a\mu - \frac{2}{3}\pi a^3 \rho \frac{d}{dt} - 6\pi a\mu \int_0^t \frac{a}{\sqrt{\pi\nu}} \frac{d\tau}{\sqrt{t-\tau}} \frac{d}{d\tau},$$

$$\mathbf{F}(t) = B\mathbf{U}(t),$$

$\nu = \mu/\rho$, μ being the viscosity, and ρ — the density of the fluid.

The first term represents an instantaneous quasi-steady Stokes drag. Two further contributions to the drag appear: $-\frac{2}{3}\pi a^3 \rho \frac{d}{dt} \mathbf{U}(t)$, the virtual mass contribution, and the so-called Basset force depending on the kernel $(t-\tau)^{-1/2}$. Hence the force exerted on the sphere is not proportional to its velocity, but it depends as well on its acceleration, and on the history of its motion.

Studies on the unsteady Stokes equations have been recently undertaken by, for example, CHOW, HERMANS [3], MAZUR, BEDEAUX [4], LEICHTBERG, WEINBAUM, PFEFFER, GLUCKMAN [5], in spite of the fact that some questions concerning the range of validity of these equations still remain unsolved [6, 7].

One of the questions discussed recently in terms of linear unsteady hydrodynamics was the generalization of the drag force exerted on a sphere, to include an arbitrary time-dependent fluid velocity. This question was considered by MAZUR and BEDEAUX [4]. They obtained the so-called Faxen's relations expressing the force exerted on the sphere in terms of the unsteady velocity of the sphere, and the unsteady fluid velocity in the absence of the sphere. The novelty of their method consists in replacing the sphere in the flow by the so-called induced forces. To calculate $\mathbf{F}(t)$, they applied the Fourier transform with respect to time. The drag $\mathbf{F}(\omega)$ obtained by Mazur and Bedeaux, is equal to

$$(1.2) \quad \mathbf{F}(\omega) = -6\pi a\mu \left[\left(1 + \alpha a + \frac{1}{9} \alpha^2 a^2 \right) \mathbf{U}(\omega) - (1 + \alpha a) \mathbf{v}_0^s(\omega) - \frac{1}{3} \alpha^2 a^2 \mathbf{v}_0^v(\omega) \right],$$

$$\alpha \equiv (-i\omega/\nu)^{1/2}, \quad \text{Re } \alpha > 0,$$

ω being the frequency, and $\mathbf{v}_0(\mathbf{r}, \omega)$ — the fluid velocity in the absence of the sphere. The velocity $\mathbf{v}_0(\mathbf{r}, \omega)$ appears in the form of $\bar{\mathbf{v}}_0^s(\omega)$, i.e. after surface averaging, and in the form of $\bar{\mathbf{v}}_0^v(\omega)$, i.e. after volume averaging:

$$(1.3) \quad \bar{\mathbf{v}}_0^s(\omega) = \frac{1}{4\pi a^2} \int_{|\mathbf{r}|=a} \mathbf{v}_0(\mathbf{r}, \omega) dS, \quad \bar{\mathbf{v}}_0^v(\omega) = \frac{3}{4\pi a^3} \int_{|\mathbf{r}| \leq a} \mathbf{v}_0(\mathbf{r}, \omega) d\mathbf{r}.$$

The dependence of the drag force on time can be described explicitly by the following relation:

$$(1.4) \quad \mathbf{F}(t) = B \{ \mathbf{U}(t) - \bar{\mathbf{v}}_0^s(t) \} + \frac{2}{3} \pi a^3 \rho \frac{d}{dt} (\bar{\mathbf{v}}_0^v(t) - \bar{\mathbf{v}}_0^s(t)) + \frac{4}{3} \pi a^3 \rho \frac{d}{dt} \bar{\mathbf{v}}_0^v(t).$$

The approach of Mazur and Bedeaux has been subsequently extended, for example, to the case of compressible fluids [8, 9], a torque exerted on the sphere [10], slip boundary conditions on the sphere surface [11, 12]. In all those papers the Fourier transform method was used. The aim of the present paper is to discuss the influence of the initial condition on the viscous resistance of a sphere. It seems that the Green function which depends explicitly on the time variable and gives the solution in a closed form can be a most useful tool for general initial conditions.

2. Governing equations

The governing equations, describing the motion of the fluid in the presence of an immersed sphere, are taken in the following form:

the equation of motion

$$(2.1) \quad \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{v} + \mathbf{f}^{\text{ext}} + \mathbf{f},$$

the equation of continuity

$$(2.2) \quad \nabla \cdot \mathbf{v} = 0,$$

with the initial condition

$$(2.3) \quad \mathbf{v}(\mathbf{r}, 0) = \boldsymbol{\xi}(\mathbf{r}).$$

To obtain the torque exerted on the sphere, the equation describing the conservation of angular momentum will be used:

$$(2.4) \quad \frac{\partial}{\partial t} (\mathbf{r} \times \mathbf{v}) = -\mathbf{r} \times \frac{1}{\rho} \nabla p + \mathbf{r} \times \nu \Delta \mathbf{v} + \mathbf{r} \times \mathbf{f}^{\text{ext}} + \mathbf{r} \times \mathbf{f}.$$

The force $\mathbf{f}(\mathbf{r}, t)$ per unit volume, appearing in the linear momentum and in the angular momentum equations, represents the presence of the sphere in the flow. It is assumed also that there is an external force $\mathbf{f}^{\text{ext}}(\mathbf{r}, t)$ per unit volume acting on the fluid. Without loss of generality it can be assumed that $\mathbf{f}(\mathbf{r}, t)$ acts inside and on the surface of the sphere, whereas $\mathbf{f}^{\text{ext}}(\mathbf{r}, t)$ acts only in the fluid [4].

Using the Green's function T_{ij}, Q_j , the formal solution to these equations become [13]

$$(2.5) \quad \begin{aligned} v_i(\mathbf{r}, t) &= v_{0i}(\mathbf{r}, t) + \int_0^t d\tau \int_{E_3} d\mathbf{r}' T_{ij}(\mathbf{r}-\mathbf{r}', t-\tau) f_j(\mathbf{r}', \tau), \\ p(\mathbf{r}, t) &= p_0(\mathbf{r}, t) + \int_0^t d\tau \int_{E_3} d\mathbf{r}' Q_j(\mathbf{r}-\mathbf{r}', t-\tau) f_j(\mathbf{r}', \tau), \end{aligned}$$

where E_3 denotes the three-dimensional space.

From the angular momentum equation, the expression for $\mathbf{r} \times \mathbf{v}(\mathbf{r}, t)$ can be derived (comp. [10]):

$$(2.6) \quad \mathbf{r} \times \mathbf{v}(\mathbf{r}, t) = [\mathbf{r} \times \mathbf{v}]_0(\mathbf{r}, t) + \int_0^t d\tau \int_{E_3} d\mathbf{r}' \Gamma(\mathbf{r}-\mathbf{r}', t-\tau) \left[-\frac{1}{\rho} \mathbf{r}' \times (\nabla p - \nabla p_0) + \mathbf{r}' \times \mathbf{f}(\mathbf{r}', \tau) \right].$$

The characteristic feature of these expressions is that the influence of the immersed sphere on the fluid velocity and pressure is given explicitly (by terms containing the force $\mathbf{f}(\mathbf{r}, t)$). These expressions have the usual form of the convolution integrals. The fundamental singular solutions used above are expressed in terms of the $\Gamma(\mathbf{r}, t)$ function:

$$(2.7) \quad T_{ij}(\mathbf{r}, t) = \Gamma(\mathbf{r}, t) \delta_{ij} + \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \int_{E_3} \frac{\Gamma(\mathbf{r}', t) d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}, \quad i, j = 1, 2, 3,$$

$$Q_j(\mathbf{r}, t) = -\frac{1}{4\pi} \delta(t) \frac{\partial}{\partial x_j} \frac{1}{|\mathbf{r}|}, \quad |\mathbf{r}|^2 = x_1^2 + x_2^2 + x_3^2,$$

$\Gamma(\mathbf{r}, t)$ representing the fundamental solution of the heat equation,

$$\Gamma(\mathbf{r}, t) = (4\pi\nu t)^{-3/2} \exp[-|\mathbf{r}|^2/4\nu t].$$

These singular solutions satisfy

$$(2.8) \quad \frac{\partial T_{ij}}{\partial t} - \nu \Delta T_{ij} + \frac{1}{\rho} \frac{\partial Q_j}{\partial x_i} = \frac{1}{\rho} \delta_{ij} \delta(t) \delta(r),$$

$$\sum_{i=1}^3 \partial T_{ij} / \partial x_i = 0.$$

The fluid velocity in the absence of the sphere $\mathbf{v}_0(\mathbf{r}, t)$ consists of two terms:

$$(2.9) \quad \mathbf{v}_0(\mathbf{r}, t) = \mathbf{v}_1(\mathbf{r}, t) + \mathbf{v}_2(\mathbf{r}, t),$$

$\mathbf{v}_1(\mathbf{r}, t)$ — generated by the external force $\mathbf{f}^{\text{ext}}(\mathbf{r}, t)$,

$$\mathbf{v}_{1i}(\mathbf{r}, t) = \int_0^t d\tau \int_{E_3} d\mathbf{r}' T_{ij}(\mathbf{r} - \mathbf{r}', t - \tau) f_j^{\text{ext}}(\mathbf{r}', \tau),$$

and $\mathbf{v}_2(\mathbf{r}, t)$ — due to the initial distribution of velocity $\boldsymbol{\xi}(\mathbf{r})$,

$$\mathbf{v}_2(\mathbf{r}, t) = \int_{E_3} d\mathbf{r}' \Gamma(\mathbf{r} - \mathbf{r}', t) \boldsymbol{\xi}(\mathbf{r}').$$

If the function $\boldsymbol{\xi}(\mathbf{r})$ is continuous and bounded for each time interval $0 \leq t \leq T$, then the estimate holds:

$$|\mathbf{v}_2(\mathbf{r}, t)| \leq \sup |\boldsymbol{\xi}(\mathbf{r}')|, \quad t > 0.$$

Similarly, the vector product $[\mathbf{r} \times \mathbf{v}]_0$ can be split into two terms:

$$(2.10) \quad [\mathbf{r} \times \mathbf{v}]_0(\mathbf{r}, t) = [\mathbf{r} \times \mathbf{v}]_1(\mathbf{r}, t) + [\mathbf{r} \times \mathbf{v}]_2(\mathbf{r}, t),$$

$$[\mathbf{r} \times \mathbf{v}]_1(\mathbf{r}, t) = \int_0^t d\tau \int_{E_3} d\mathbf{r}' \Gamma(\mathbf{r} - \mathbf{r}', t - \tau) \left[-\frac{1}{\rho} \mathbf{r}' \times \nabla p_0 + \mathbf{r}' \times \mathbf{f}^{\text{ext}} - 2\nu \text{rot } \mathbf{v}_1 \right],$$

and

$$[\mathbf{r} \times \mathbf{v}]_2(\mathbf{r}, t) = \int_{E_3} d\mathbf{r}' \Gamma(\mathbf{r} - \mathbf{r}', t) (\mathbf{r} - \mathbf{r}') \times \boldsymbol{\xi}(\mathbf{r}').$$

3. The force and torque exerted on the sphere

Using the equation of motion, the force exerted on the sphere can be expressed through the velocity field and the force $\mathbf{f}(\mathbf{r}, t)$ averaged over the sphere volume:

$$(3.1) \quad \mathbf{F}(t) = - \int_S \mathbf{P} \cdot \mathbf{n} dS = \varrho \frac{d}{dt} \int_{|\mathbf{r}| \leq a} \mathbf{v}(\mathbf{r}, t) d\mathbf{r} - \varrho \int_{|\mathbf{r}| \leq a} \mathbf{f}(\mathbf{r}, t) d\mathbf{r},$$

$$P_{ij} = p\delta_{ij} - 2\mu \left(\frac{\partial \mathbf{v}}{\partial \mathbf{r}} \right)_{ij}, \quad \left(\frac{\partial \mathbf{v}}{\partial \mathbf{r}} \right)_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i}),$$

where S — the sphere surface, \mathbf{P} — the stress tensor, \mathbf{n} — the unit normal vector directed outward of the sphere.

From the angular momentum equation an analogous relation can be obtained for the torque acting on the sphere:

$$(3.2) \quad \mathbf{T}(t) = - \int_S \mathbf{r} \times \mathbf{P} \cdot \mathbf{n} dS = \varrho \frac{d}{dt} \int_{|\mathbf{r}| \leq a} \mathbf{r} \times \mathbf{v}(\mathbf{r}, t) d\mathbf{r} - \varrho \int_{|\mathbf{r}| \leq a} \mathbf{r} \times \mathbf{f}(\mathbf{r}, t) d\mathbf{r}.$$

The flow of the fluid and the motion of the sphere are coupled by means of the boundary conditions assumed on the surface of the sphere. Here the slip boundary conditions are considered:

$$(3.3) \quad \mathbf{v}(\mathbf{r}, t) \cdot \mathbf{n} = \mathbf{U}(t) \cdot \mathbf{n}, \quad |\mathbf{r}| = a,$$

$$(3.4) \quad (1 - \mathbf{nn}) \left(\mathbf{v}(\mathbf{r}, t) - 2\lambda \frac{\partial \mathbf{v}}{\partial \mathbf{r}} \mathbf{n} \right) = (1 - \mathbf{nn}) (\mathbf{U}(t) + \boldsymbol{\Omega}(t) \times \mathbf{r}), \quad |\mathbf{r}| = a.$$

Here λ is the slip coefficient, $\lambda \in [0, \infty)$, and $\boldsymbol{\Omega}(t)$ is the angular velocity of the sphere. The surface of the sphere is given simply by $|\mathbf{r}| = a$ due to the fact that the set of coordinate axes is fixed at the centre of a sphere. This lack of a time dependence of the sphere position expresses the assumed linearization of the problem.

Following MAZUR's approach [11], one can average the boundary conditions over the sphere surface. Hence, averaging the expression (3.3) and using the Gauss theorem to change from the surface integral to the volume integral, one obtains

$$(3.5) \quad \bar{\mathbf{v}}^p(t) = \mathbf{U}(t).$$

From the second boundary conditions (3.4), it follows:

$$(3.6) \quad \bar{\mathbf{v}}^s(t) = \mathbf{U}(t) + \frac{\lambda}{a + 2\lambda} \left(r \frac{\partial}{\partial r} \bar{\mathbf{v}}^s(r) \right)_{r=a},$$

for the translational motion and for the rotational motion:

$$(3.7) \quad \overline{\mathbf{r} \times \mathbf{v}}^s(t) = \frac{2}{3} a^2 \boldsymbol{\Omega}(t) + \frac{\lambda}{a} \left(-2\mathbf{r} \times \bar{\mathbf{v}}^s + \left(r \frac{\partial}{\partial r} \overline{\mathbf{r} \times \mathbf{v}}^s(r) \right)_{r=a} \right),$$

where $\bar{\mathbf{v}}^{s(r)}$ denotes the velocity field averaged over the surface of the sphere of radius r

$$\left(\bar{\mathbf{v}}^{s(r)}(t) = \frac{1}{4\pi r^2} \int_{|\mathbf{R}|=r} \mathbf{v}(\mathbf{R}, t) dS \right).$$

From the previous relations (see App. A, B), the force and torque exerted on the sphere is deduced in terms of $\mathbf{v}_p(\mathbf{r}, t)$, the perturbation of the fluid velocity due to the presence of the sphere:

$$(3.8) \quad \begin{aligned} \mathbf{v}_p(\mathbf{r}, t) &= \mathbf{v}(\mathbf{r}, t) - \mathbf{v}_0(\mathbf{r}, t), \\ [\mathbf{r} \times \mathbf{v}]_p(\mathbf{r}, t) &= [\mathbf{r} \times \mathbf{v}](\mathbf{r}, t) - [\mathbf{r} \times \mathbf{v}]_0(\mathbf{r}, t). \end{aligned}$$

The force and the torque are

$$(3.9) \quad \mathbf{F}(t) - \frac{4}{3} \pi a^3 \rho \frac{d\bar{\mathbf{v}}^v}{dt} = B \{\bar{\mathbf{v}}_p^s\} + \frac{2}{3} \pi a^3 \rho \frac{d}{dt} \bar{\mathbf{v}}_p^s - 2\pi a^3 \rho \frac{d}{dt} \bar{\mathbf{v}}_p^v,$$

$$(3.10) \quad \mathbf{T}(t) - \frac{4}{3} \pi a^3 \rho \frac{d}{dt} \overline{\mathbf{r} \times \mathbf{v}}^v = C \{[\overline{\mathbf{r} \times \mathbf{v}}]_p^s\} - \frac{4}{3} \pi a^3 \rho \frac{d}{dt} [\overline{\mathbf{r} \times \mathbf{v}}]_p^v,$$

where two terms, appearing on the left hand side, describe the effect of the inertia of the fluid displaced by the immersion of the rigid sphere.

The C operator is of the following form:

$$(3.11) \quad C = -12\pi a \mu - 12\pi a \mu \int_0^t d\tau \left[\frac{a}{3\sqrt{\pi\nu}} \frac{1}{\sqrt{t-\tau}} - \frac{1}{3} \exp \frac{\nu(t-\tau)}{a^2} \operatorname{erfc} \sqrt{\frac{\nu(t-\tau)}{a^2}} \right] \frac{d}{d\tau}.$$

Hence the expression for the torque has a similar structure as in the case of the force. The C operator contains an instantaneous quasi-steady Stokes term and a term involving the history of the motion. This operator acting on $\frac{2}{3} a^2 \Omega(t)$ gives the torque exerted on the sphere when the fluid is at rest at infinity, and on the surface of the sphere the stick boundary conditions are assumed.

The above formulae do not have as yet a form of the Faxen's relations. To deduce that form the perturbed fluid velocity will be presented in terms of $\mathbf{U}(t)$, and $\mathbf{v}_0(\mathbf{r}, t)$ (see App. A, B). The appropriate relations are as follows:

$$(3.12) \quad \bar{\mathbf{v}}_p^s = \mathbf{U} - \bar{\mathbf{v}}_0^s + \frac{\lambda}{a+2\lambda} \left[\frac{\mathbf{F}}{6\pi\mu a} + \left(r \frac{\partial}{\partial r} \bar{\mathbf{v}}_0^{s(r)} \right)_{r=a} + \frac{a^2}{9\nu} \frac{d}{dt} (\mathbf{U} - 3\bar{\mathbf{v}}_0^v) \right],$$

and

$$(3.13) \quad \begin{aligned} [\overline{\mathbf{r} \times \mathbf{v}}]_p^s &= \frac{2}{3} a^2 \Omega - [\overline{\mathbf{r} \times \mathbf{v}}]_0^s + \frac{\lambda}{a} \left[-2[\overline{\mathbf{r} \times \mathbf{v}}]_0^s + \frac{\mathbf{T}}{4\pi\mu a} \right. \\ &\quad \left. + \left(r \frac{\partial}{\partial r} [\overline{\mathbf{r} \times \mathbf{v}}]_0^{s(r)} \right)_{r=a} - \frac{a^2}{3\nu} \frac{d}{dt} [\overline{\mathbf{r} \times \mathbf{v}}]_0^v \right]. \end{aligned}$$

It can be seen that in the case of no-tangential slip boundary conditions the last term in both of these expressions vanish.

Finally, the expression for the force $\mathbf{F}(t)$ assumes the following form:

$$(3.14) \quad \begin{aligned} \mathbf{F}(t) + \frac{a\lambda}{(a+3\lambda)\sqrt{\pi\nu}} \int_0^t \frac{d\tau}{\sqrt{t-\tau}} \frac{d\mathbf{F}}{d\tau} &= -\frac{2}{3} \pi a^3 \rho \frac{d}{dt} [\mathbf{U} - \bar{\mathbf{v}}_0^v] + \frac{4}{3} \pi a^3 \rho \frac{d}{dt} \bar{\mathbf{v}}_0^s + \\ &\quad -6\pi\mu a \left[\frac{a+2\lambda}{a+3\lambda} (\mathbf{U} - \bar{\mathbf{v}}_0^s) + \frac{\lambda}{a+3\lambda} \left(r \frac{\partial}{\partial r} \bar{\mathbf{v}}_0^{s(r)} \right)_{r=a} \right] + \end{aligned}$$

$$-6a^2 \sqrt{\pi\mu\varrho} \int_0^t \frac{d\tau}{\sqrt{t-\tau}} \frac{d}{d\tau} \left[\frac{a+2\lambda}{a+3\lambda} (\mathbf{U} - \bar{\mathbf{v}}_0^s) + \frac{\lambda}{a+3\lambda} \left[\frac{a^2}{9\nu} \frac{d}{d\tau} (U - 3\bar{v}_0^s) + \left(r \frac{\partial}{\partial r} \bar{\mathbf{v}}_0^{s(r)} \right)_{r=a} \right] \right].$$

It consists of two known unsteady terms, the generalized instantaneous quasi-Stokes term and the Basset terms involving the history of the motion both of the sphere and the fluid in the absence of the sphere. In the case of the torque, the appropriate formula has a similar structure:

$$(3.15) \quad \mathbf{T}(t) + \frac{3\lambda}{a+3\lambda} \int_0^t d\tau K(t-\tau) \frac{d}{d\tau} \mathbf{T} = \frac{4}{3} \pi a^3 \varrho \frac{d}{dt} [\mathbf{r} \times \mathbf{v}]_0^v - \frac{12\pi\mu a}{a+3\lambda} \left[\frac{2}{3} a^3 \boldsymbol{\Omega} - (a+2\lambda) [\mathbf{r} \times \mathbf{v}]_0^s + \lambda \left(r \frac{\partial}{\partial r} [\mathbf{r} \times \mathbf{v}]_0^{s(r)} \right)_{r=a} \right] - \frac{12\pi\mu a}{a+3\lambda} \int_0^t d\tau K(t-\tau) \frac{d}{d\tau} \left[\frac{2}{3} a^3 \boldsymbol{\Omega} - (a+2\lambda) [\mathbf{r} \times \mathbf{v}]_0^s + \lambda \left(r \frac{\partial}{\partial r} [\mathbf{r} \times \mathbf{v}]_0^{s(r)} \right)_{r=a} - \frac{\lambda a^2}{3\nu} \frac{d}{d\tau} [\mathbf{r} \times \mathbf{v}]_0^v \right],$$

where

$$K(t-\tau) = \frac{a}{3\sqrt{\pi\nu}} \frac{1}{\sqrt{t-\tau}} - \frac{1}{3} \exp \frac{\nu(t-\tau)}{a^2} \operatorname{erfc} \sqrt{\frac{\nu(t-\tau)}{a^2}}.$$

The influence of the initial velocity field is hidden in $\mathbf{v}_0(\mathbf{r}, t)$ and respectively $[\mathbf{r} \times \mathbf{v}]_0(\mathbf{r}, t)$ (see the formulae (2.9) and (2.10)).

An asymptotic behaviour in the limit $t \rightarrow \infty$ of the quantities (3.14) and (3.15) is interesting if one considers the approach to the steady case. In the fluid at rest at infinity, for the no-slip boundary conditions, the classical result is recovered [14]:

$$(3.16) \quad \mathbf{F}(t) \xrightarrow{t \rightarrow \infty} -6\pi a \mu \mathbf{U} \left[1 + \frac{a}{\sqrt{\pi\nu t}} + \dots \right],$$

$$\mathbf{T}(t) \xrightarrow{t \rightarrow \infty} -8\pi a^3 \mu \boldsymbol{\Omega} \left[1 + \frac{a^3}{6\sqrt{\pi(\nu t)^{3/2}}} + \dots \right].$$

Hence the force and the torque have a different asymptotic behaviour.

4. Conclusions

It is thus shown that the Green's function (depending explicitly on the time variable) method is convenient to derive the force and the torque acting on a sphere. These quantities were deduced also without solving the full hydrodynamic problem; this means, without

finding explicitly the fluid velocity and the pressure field. The effect due to the initial velocity field is accounted for.

The initial velocity field influences the virtual mass contribution, the instantaneous quasi-Stokes term, as well as the Basset force, depending on the history of the motion. Its effect on the force and the torque exerted on the sphere depends on the relation between the velocity field $\mathbf{v}_1(\mathbf{r}, t)$ generated by the external force and the velocity $\mathbf{v}_2(\mathbf{r}, t)$ — generated only by the initial distribution of fluid velocity. In the formulae (3.14) and (3.15), giving the force and the torque, both velocity fields enter in a similar way.

Appendix A

The velocity field in the presence of the sphere is

$$(A.1) \quad \mathbf{v}_i(\mathbf{r}, t) = \mathbf{v}_{0i}(\mathbf{r}, t) + \int_0^t d\tau \int_{|\mathbf{r}'| \leq a} d\mathbf{r}' \left\{ \Gamma(\mathbf{r} - \mathbf{r}', t - \tau) \delta_{ij} + \frac{\partial^2}{\partial x_i \partial x_j} \left[\frac{1}{2\pi|\mathbf{r} - \mathbf{r}'|} \theta \left(\frac{|\mathbf{r} - \mathbf{r}'|}{\sqrt{4\nu(t - \tau)}} \right) \right] \right\} f_j(\mathbf{r}', \tau),$$

where

$$\theta \left(\frac{|\mathbf{r}|}{\sqrt{4\nu t}} \right) = \int_0^{\frac{|\mathbf{r}|^2}{4\nu t}} e^{-\eta^2} d\eta.$$

Averaging this quantity over the surface and the volume of the sphere, one obtains

$$(A.2) \quad \bar{v}_i^s(t) = \bar{v}_{0i}^s(t) + \frac{1}{4\pi^{3/2}} \int_0^t d\tau \int_{|\mathbf{r}'| \leq a} d\mathbf{r}' \left\{ \frac{\alpha}{ar'} [e^{-\kappa_1^2} - e^{-\kappa_2^2}] \delta_{ij} + \frac{\partial^2}{\partial x_i \partial x_j} \left[\frac{a+r'}{ar'} \theta(\kappa_2) - \frac{a-r'}{ar'} \theta(\kappa_1) - \frac{1}{2\alpha ar'} (e^{-\kappa_1^2} - e^{-\kappa_2^2}) \right] \right\} f_j(\mathbf{r}', \tau),$$

and, respectively,

$$(A.3) \quad \bar{v}_i^v(t) = \bar{v}_{0i}^v(t) + \frac{1}{4\pi^{3/2}} \int_0^t d\tau \int_{|\mathbf{r}'| \leq a} d\mathbf{r}' \left\{ \left[\frac{3}{a^3} \theta(\kappa_1) + \frac{3}{a^3} \theta(\kappa_2) - \frac{3}{2\alpha a^3 r'} (e^{-\kappa_1^2} - e^{-\kappa_2^2}) \right] \delta_{ij} + \frac{\partial^2}{\partial x_i \partial x_j} \left[\frac{1}{r'} (\theta(\kappa_2) - \theta(\kappa_1)) + \left(\frac{3}{2a} - \frac{r'^2}{2a^3} - \frac{3}{4a^3 \alpha^2} \right) (\theta(\kappa_1) + \theta(\kappa_2)) + \frac{1}{4\alpha a^2} (e^{-\kappa_1^2} + e^{-\kappa_2^2}) + \left(\frac{1}{4\alpha^3 a^3 r'} - \frac{1}{2\alpha ar'} + \frac{r'}{4\alpha a^3} \right) (e^{-\kappa_1^2} - e^{-\kappa_2^2}) \right] \right\} f_j(\mathbf{r}', \tau),$$

where the following notation is used:

$$r' = |\mathbf{r}'|, \quad \alpha = 1/\sqrt{4\nu(t-\tau)},$$

$$\kappa_1 = \alpha(a-r'), \quad \kappa_2 = \alpha(a+r').$$

Next, taking into account the facts that the B operator contains the time derivative of the velocity of the sphere, and that the relation (3.5) holds, the time derivatives of Eqs. (A.2) and (A.3) were calculated. They are equal to

$$(A.4) \quad \frac{d}{dt} \bar{v}_i^s(t) = \frac{d}{dt} \bar{v}_{0i}^s(t) + \frac{\nu}{\pi^{3/2}} \int_0^t d\tau \int_{r' \leq a} d\mathbf{r}' \left\{ \left[e^{-\kappa_1^2} \left[-\frac{\alpha^3}{2ar'} + \frac{\alpha^5(a-r')^2}{ar'} \right] \right. \right.$$

$$\left. \left. + e^{-\kappa_2^2} \left[\frac{\alpha^3}{2ar'} - \frac{\alpha^5(a+r')^2}{ar'} \right] \right] \delta_{ij} + \frac{\partial^2}{\partial x_i \partial x_j} \left[-\frac{\alpha}{4ar'} (e^{-\kappa_1^2} - e^{-\kappa_2^2}) \right] \right\} f_j(\mathbf{r}', \tau),$$

and, respectively,

$$(A.5) \quad \frac{d}{dt} \bar{v}_i^p(t) = \frac{d}{dt} \bar{v}_{0i}^p(t) + \frac{1}{2\pi a^3} \int_{r' \leq a} d\mathbf{r}' f_i(\mathbf{r}', t)$$

$$+ \frac{\nu}{\pi^{3/2}} \int_0^t d\tau \int_{r' \leq a} d\mathbf{r}' \left\{ \left[e^{-\kappa_1^2} \left(-\frac{3\alpha}{4a^3 r'} - \frac{3\alpha^3(a-r')}{2a^2 r'} \right) + e^{-\kappa_2^2} \left(\frac{3\alpha}{4a^3 r'} + \frac{3\alpha^3(a+r')}{2a^2 r'} \right) \right] \delta_{ij} \right.$$

$$\left. + \frac{\partial^2}{\partial x_i \partial x_j} \left[-\frac{3}{4a^3} (\theta(\kappa_1) + \theta(\kappa_2)) + \frac{3}{8\alpha a^3 r'} (e^{-\kappa_1^2} - e^{-\kappa_2^2}) \right] \right\} f_j(\mathbf{r}', \tau).$$

The induced force averaged over the volume of the sphere can be eliminated between Eqs. (A.2) and (A.5) to give

$$(A.6) \quad \varrho \int_{|\mathbf{r}| \leq a} d\mathbf{r} \mathbf{f}(\mathbf{r}, t) = 2\pi a^3 \varrho \frac{d}{dt} \bar{\mathbf{v}}_p^s + 6\pi a \mu \bar{\mathbf{v}}_p^s + 6a^2 \sqrt{\pi \mu \varrho} \int_0^t \frac{d\tau}{\sqrt{t-\tau}} \frac{d}{d\tau} \bar{\mathbf{v}}_p^s.$$

The need for such an expression for $\int d\mathbf{r} \mathbf{f}(\mathbf{r}, t)$ was again stimulated by the form of the B operator. The force acting on the sphere, according to Eq. (3.1), is equal to

$$(A.7) \quad \mathbf{F}(t) = -\frac{2}{3} \pi a^3 \varrho \frac{d}{dt} \mathbf{U} + 2\pi a^3 \varrho \frac{d}{dt} \bar{\mathbf{v}}_0^s - 6\pi \mu a \bar{\mathbf{v}}_p^s - 6a^2 \sqrt{\pi \mu \varrho} \int_0^t \frac{d\tau}{\sqrt{t-\tau}} \frac{d}{d\tau} \bar{\mathbf{v}}_p^s.$$

In the formula above it was taken into account that

$$(A.8) \quad \mathbf{U}(t) = \bar{\mathbf{v}}^s(t),$$

which follows from the boundary condition (3.3).

To present the force $\mathbf{F}(t)$ in the form of the Faxén relation, the following formula (resulting from the boundary conditions (3.4), and derived in [11]) is applied:

$$(A.9) \quad \bar{\mathbf{v}}^s(t) = \mathbf{U}(t) + \frac{\lambda}{a+2\lambda} \left(r \frac{\partial}{\partial r} \bar{\mathbf{v}}^s(r)(t) \right)_{r=a}.$$

In view of Eq. (A.5) one obtains

$$(A.10) \quad \left(r \frac{\partial}{\partial r} \bar{\mathbf{v}}_p^{s(r)} \right)_{r=a} = \frac{\mathbf{F}}{6\pi\mu a} + \frac{a^2}{9\nu} \frac{d}{dt} (\mathbf{U} - 3\bar{\mathbf{v}}_0^s),$$

and hence the expression (A.9) can be rewritten in the form

$$(A.11) \quad \bar{\mathbf{v}}_p^s = \mathbf{U} - \bar{\mathbf{v}}_0^s + \frac{\lambda}{a + 2\lambda} \left[\frac{\mathbf{F}}{6\pi\mu a} + \frac{a^2}{9\nu} \frac{d}{dt} (\mathbf{U} - 3\bar{\mathbf{v}}_0^s) + \left(r \frac{\partial}{\partial r} \bar{\mathbf{v}}_p^{s(r)} \right)_{r=a} \right].$$

Substituting $\bar{\mathbf{v}}_p^s$ to Eq. (A.7), one obtains Eq. (3.14).

Appendix B

Using the momentum and the angular momentum equations one obtains

$$(B.1) \quad \mathbf{r} \times \mathbf{v}(\mathbf{r}, t) = [\mathbf{r} \times \mathbf{v}]_0(\mathbf{r}, t) + \int_0^t d\tau \int_{E_3} d\mathbf{r}' T(\mathbf{r} - \mathbf{r}', t - \tau) \left[-\frac{\mathbf{r}'}{\rho} \times (\nabla p - \nabla p_0) + \mathbf{r} \times \mathbf{f}(\mathbf{r}', \tau) \right].$$

As previously, to calculate the torque exerted on the sphere, the following quantities will be applied:

$$(B.2) \quad \overline{\mathbf{r} \times \mathbf{v}^s}(t) = \overline{[\mathbf{r} \times \mathbf{v}]_0^s}(t) + \frac{1}{4\pi^{3/2}} \int_0^t d\tau \int_{E_3} d\mathbf{r}' \frac{\mathbf{r}'}{|\mathbf{r}'|} \left[-\kappa_1 \left(\frac{\alpha}{r'} - \frac{1}{2\alpha ar'^2} \right) + e^{-\kappa_2^2} \left(\frac{\alpha}{r'} + \frac{1}{2\alpha ar'^2} \right) \right] \times \mathbf{f}(\mathbf{r}', \tau),$$

and respectively

$$(B.3) \quad \overline{\mathbf{r} \times \mathbf{v}^v}(t) = \overline{[\mathbf{r} \times \mathbf{v}]_0^v} + \frac{1}{4\pi^{3/2}} \int_0^t d\tau \int_{E_3} d\mathbf{r}' \frac{\mathbf{r}'}{|\mathbf{r}'|} \left[\frac{3r'}{a^3} (\theta(\kappa_1) + \theta(\kappa_2)) + (e^{-\kappa_1^2} - e^{-\kappa_2^2}) \left(\frac{3}{4a^3 r'^2 \alpha^3} - \frac{3}{2a^3 \alpha} \right) - (e^{-\kappa_1^2} + e^{-\kappa_2^2}) \frac{3}{2a^2 r' \alpha} \right] \times \mathbf{f}(\mathbf{r}', \tau).$$

As it was discussed by HILLS [10], the pressure gradient does not contribute to the above expressions.

For the subsequent calculations we will also need

$$(B.4) \quad \frac{d}{dt} \overline{\mathbf{r} \times \mathbf{v}^s} = \frac{d}{dt} \overline{[\mathbf{r} \times \mathbf{v}]_0^s} + \frac{\nu}{\pi^{3/2}} \int_0^t d\tau \int_{E_3} d\mathbf{r}' \frac{\mathbf{r}'}{|\mathbf{r}'|} \left\{ e^{-\kappa_1^2} \left[\alpha^5 \frac{(a-r')^2}{r'} + \alpha^3 \frac{ar' - a^2 - r'^2}{2ar'^2} - \frac{\alpha}{4ar'^2} \right] + e^{-\kappa_2^2} \left[\alpha^5 \frac{(a+r')^2}{r'} + \alpha^3 \frac{ar' + a^2 + r'^2}{2ar'^2} + \frac{\alpha}{4ar'^2} \right] \right\} \times \mathbf{f}(\mathbf{r}', \tau),$$

and further

$$(B.5) \quad \frac{d}{dt} \overline{\mathbf{r} \times \mathbf{v}}^v = \frac{d}{dt} \overline{\mathbf{r} \times \mathbf{v}}_0^v + \frac{3}{4\pi a^3} \int_{|\mathbf{r}| \leq a} d\mathbf{r} \mathbf{r} \times \mathbf{f}(\mathbf{r}, t) \\ + \frac{\nu}{\pi^{3/2}} \int_0^t d\tau \int_{E_3} d\mathbf{r}' \frac{\mathbf{r}'}{|\mathbf{r}'|} \left\{ e^{-\kappa_1^2} \left[-\frac{3}{2} \alpha^3 \frac{a-r'}{ar'} + \alpha \frac{3a-9r'}{4a^2 r'^2} + \frac{1}{\alpha} \frac{9}{8a^3 r'^2} \right] \right. \\ \left. + e^{-\kappa_2^2} \left[-\frac{3}{2} \alpha^3 \frac{a+r'}{ar'} - \alpha \frac{3a+9r'}{4a^2 r'^2} - \frac{1}{\alpha} \frac{9}{8a^3 r'^2} \right] \right\} \times \mathbf{f}(\mathbf{r}', \tau).$$

In view of the definition (3.2) and using Eq. (B.5), the torque exerted on the sphere can be presented in the following form:

$$(B.6) \quad \mathbf{T}(t) = \frac{4}{3} \pi a^3 \rho \frac{d}{dt} \overline{\mathbf{r} \times \mathbf{v}}_0^v + \frac{\mu}{2\sqrt{\pi}} \int_0^t d\tau \int_{|\mathbf{r}'| \leq a} d\mathbf{r}' \frac{\mathbf{r}'}{|\mathbf{r}'|} \left\{ (e^{-\kappa_1^2} + e^{-\kappa_2^2}) \left(-\frac{6a\alpha}{r'} \right. \right. \\ \left. \left. - \frac{4a^3 \alpha^3}{r'} \right) + (e^{-\kappa_1^2} - e^{-\kappa_2^2}) \left(\frac{3}{r'^2 \alpha} + \frac{2a^2 \alpha}{r'^2} + 4a^2 \alpha^3 \right) \right\} \times \mathbf{f}(\mathbf{r}', \tau).$$

Hence, taking into account Eqs. (B.2) and (B.4), it can be shown that

$$(B.7) \quad \mathbf{T}(t) = \frac{4}{3} \pi a^3 \rho \frac{d}{dt} \overline{\mathbf{r} \times \mathbf{v}}_0^v - 12\pi a \mu \overline{\mathbf{r} \times \mathbf{v}}_p^s \\ - 12\pi a \mu \int_0^t d\tau \left[\frac{a}{3\sqrt{\pi\nu}} \frac{1}{\sqrt{t-\tau}} - \frac{1}{3} \exp \frac{\nu(t-\tau)}{a^2} \operatorname{erfc} \sqrt{\frac{\nu(t-\tau)}{a^2}} \right] \frac{d}{d\tau} \overline{\mathbf{r} \times \mathbf{v}}_p^s$$

The structure of the above formula is similar to that for the torque acting on the sphere in the fluid at rest at infinity [15]. The next step was to calculate $\overline{\mathbf{r} \times \mathbf{v}}_p^s$, applying the bound ary conditions (3.4). One obtains

$$(B.8) \quad \overline{\mathbf{r} \times \mathbf{v}}^s = \frac{2}{3} a^2 \boldsymbol{\Omega} + \frac{\lambda}{a} \left(-2\overline{\mathbf{r} \times \mathbf{v}}^s + \left(r \frac{\partial}{\partial r} \overline{\mathbf{r} \times \mathbf{v}}^{s(r)} \right)_{r=a} \right).$$

The derivative $\left(r \frac{\partial}{\partial r} \overline{\mathbf{r} \times \mathbf{v}}^{s(r)} \right)_{r=a}$ can be written down in the following form:

$$(B.9) \quad \left(r \frac{\partial}{\partial r} \overline{\mathbf{r} \times \mathbf{v}}^{s(r)} \right)_{r=a} = \left(r \frac{\partial}{\partial r} \overline{\mathbf{r} \times \mathbf{v}}_0^{s(r)} \right)_{r=a} + \frac{\mathbf{T}}{4\pi\mu a} + 2\overline{\mathbf{r} \times \mathbf{v}}_p^s - \frac{a^2}{3\nu} \frac{d}{dt} \overline{\mathbf{r} \times \mathbf{v}}_0^v.$$

Combining Eqs. (B.8) and (B.9), one gets

$$(B.10) \quad \overline{\mathbf{r} \times \mathbf{v}}_p^s = \frac{2}{3} a^2 \boldsymbol{\Omega} - \left(1 + 2 \frac{\lambda}{a} \right) \overline{\mathbf{r} \times \mathbf{v}}_0^s \\ + \frac{\lambda}{a} \left[\left(r \frac{\partial}{\partial r} \overline{\mathbf{r} \times \mathbf{v}}_0^{s(r)} \right)_{r=a} + \frac{\mathbf{T}}{4\pi\mu a} - \frac{a^2}{3\nu} \frac{d}{dt} \overline{\mathbf{r} \times \mathbf{v}}_0^v \right]$$

The above expression is substituted to Eq. (B.7) to calculate Eq. (3.15).

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