

On the dynamic spaces and on the equations of motion of nonlinear nonholonomic mechanical systems

A. SZATKOWSKI (GDAŃSK)

REGULARITY of nonlinear nonholonomic mechanical systems is discussed. A lumped mechanical system is called here the regular one, if the dynamic space of the system, i.e., the space plaited of the solution curves, equals the whole configuration space of the system — the space defined by the constraints imposed on accessible positions and velocities of the system, and the system has a well defined dynamic equation. Shrinkage of the configuration space can be easily observed among the electrical networks where one can find simple constructions of nonregular systems. However, except the lumped mechanical systems which exhibit some kind of discontinuous modes, the well-posed mechanical systems which are of practical interest are the regular systems. Thus, the conditions ensuring regularity and proved here for a broad class of mechanical systems are of special importance. It has been also shown that holonomic systems are regular, and hence, the examples of nonregular mechanical systems are among the nonholonomic systems. Hence, much attention has been paid to nonholonomic systems. A general procedure for finding the space of motion and the dynamic equation of a nonregular mechanical system is proposed. The description presented is the extension of the theory initiated in the area of electrical networks.

Dyskutowana jest regularność nieliniowych nieholonomicznych układów mechanicznych. Układ mechaniczny (o stałych skupionych) jest określany jako regularny, jeżeli jego przestrzeń dynamiczna, tzn. przestrzeń wyznaczona przez wszystkie trajektorie układu, jest identyczna z rozszerzoną przestrzenią konfiguracyjną układu, przestrzenią wyznaczoną przez więzy nałożone na dopuszczalne położenia i prędkości. W określeniu regularności wymaga się też, aby układ mechaniczny posiadał dobrze określone równanie dynamiczne. Efekt zwężenia przestrzeni konfiguracyjnej jest obserwowany wśród układów elektrycznych o stosunkowo prostej strukturze i gładkim przebiegu trajektorii. Układy mechaniczne, z wyjątkiem tych, które wykazują pewnego typu nieciągłości rozwiązań, są układami regularnymi. Podane zostały warunki zapewniające regularność dla szerokiej klasy układów. Wykazano, że układy holonomiczne są zawsze regularne. Szczególnie wiele uwagi poświęcono więc układom nieholonomicznym. Przedstawiona została ogólna metoda wyznaczania przestrzeni dynamicznej i równania ruchu nieregularnego układu mechanicznego. Podane w pracy sformułowanie stanowi rozwinięcie analogicznych rozwiązań zainicjowanych w ramach teorii sieci elektrycznych.

Обсуждается регулярность нелинейных неголономических механических систем. Механическая система (со сосредоточенными постоянными) определяется как регулярная, если ее динамическое пространство, т. э. пространство, определенное всеми траекториями системы, идентичное с расширенным конфигурационным пространством системы, пространством определенным связями, наложенными на допустимые положения и скорости. В определении регулярности требуется тоже, чтобы механическая система имела хорошо определенное динамическое уравнение. Эффект сужения конфигурационного пространства наблюдается среди электрических систем со сравнительно простой структурой и гладким ходом траектории. Механические системы, за исключением тех, которые обладают некоторого типа разрывами решений, являются регулярными системами. Приведены условия, обеспечивающие регулярность для широкого класса систем. Показано, что голономические системы всегда регулярные. Особенно много внимания посвящено же неголономическим системам. Представлен общий метод определения динамического пространства и уравнения движения нерегулярной механической системы. Приведенная в работе формулировка составляет развитие аналогичных рассуждений, начало которых приведено в рамках теории электрических сетей.

1. Introduction

IN THE PAPER lumped mechanical systems composed of a finite number of material particles $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_q$ and observed in a fixed inertial reference system $R^3 \times R_t$ are considered.

The extended configuration space (the Newton state space [6]) of a mechanical system \mathcal{A} is a subset $W_{\mathcal{A}}$ defined by the constraints in the position-velocity space $R^{3q} \times R^{3q}$ of the system. In the general case, when the constraints are nonholonomic, the extended configuration space of a mechanical system would have the structure of a fibre bundle embedded in $R^{3q} \times R^{3q}$ [3]. The configuration space of the system \mathcal{A} is the projection of the extended configuration space $W_{\mathcal{A}}$ of \mathcal{A} on the position space of \mathcal{A} [cf. [1], [6]].

The dynamic space $\mathcal{M}_{\mathcal{A}}$ of a mechanical system \mathcal{A} (the space of motion of \mathcal{A}) is a subset of the extended configuration space $W_{\mathcal{A}}$ plaited of the solution curves of the system. The system \mathcal{A} is said to be regular if the dynamic space $\mathcal{M}_{\mathcal{A}}$ of \mathcal{A} equals the entire extended configuration space of \mathcal{A} and the system has a well-defined dynamic equation on $\mathcal{M}_{\mathcal{A}}$.

Shrinkage of the (extended) configuration space can be easily observed among the electrical networks where one can find simple constructions of nonregular systems (cf. [7, 8]). However, apart from the lumped mechanical systems constructed in such a way as to exhibit some kind of discontinuous modes [5], the well-posed mechanical systems which are of practical interest are regular systems.

We concentrate our attention on regular systems in Sect. 4 where the conditions ensuring regularity of a mechanical system are proposed.

Nonregular mechanical systems are discussed in Sect. 3 where a general procedure for finding the dynamic space and the dynamic equation of a mechanical system is proposed. And in the Appendix, a general theorem is proved, the Theorem A.4, which says that the dynamic space of a mechanical system \mathcal{A} is the set union of all invariant submanifolds of the extended configuration space of \mathcal{A} . An example of a nonregular mechanical system is included in Sect. 4 of the paper.

The d'Alembert principle, in its version assuming the dual reactions algorithm [6], serves as the basis for the considerations. The concept of the d'Alembert space, introduced in Sect. 2.2, expresses in purely geometric terms the contents of the d'Alembert principle, and it is very useful when the problem of finding the dynamic space and the dynamic equation of a mechanical system is considered.

2. Basic definitions. The d'Alembert space of a lumped mechanical system

The set $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_q)$ is given, q being a natural number whose elements are named material particles. To each $\mathcal{P}_j \in \mathcal{P}, j = 1, 2, \dots, q$, a positive constant m_j is assigned, called the inert mass of the particle \mathcal{P}_j , and the quartet (x_j^1, x_j^2, x_j^3, t) of variables being the coordinates of \mathcal{P}_j in the chosen inertial reference system $R^3 \times R_t$. x_j^1, x_j^2, x_j^3 are the space coordinates of \mathcal{P}_j and t is the time coordinate.

Set $m = (m_1, m_2, \dots, m_q)$ for the masses vector.

Write $R_{\mathcal{A}}^{3q}$ (we use the symbol \mathcal{A} to denote the mechanical system) for Euclidean space of points $X = (X^1, X^2, \dots, X^{3q}) = (x_1^1, x_1^2, x_1^3, \dots, x_q^1, x_q^2, x_q^3)$ and call $R_{\mathcal{A}}^{3q}$ the position space of the mechanical system \mathcal{A} observed in the fixed inertial reference system $R^3 \times R_t$.

In ⁽¹⁾ $TR_A^{3q} \cong R_A^{3q} \times R^{3q}$, the position-velocity space of A (cf. [1, 6]), a subset W_A is given, which has the structure of differentiable submanifold of TR_A^{3q} , and which additionally has the structure of a fibre bundle $W_A = (W_A, p_X \circ l_{W_A}(\cdot), N_A)$ where N_A is a differentiable C^2 -submanifold of R_A^{3q} and $W_A = \bigcup_{X \in N_A} (X, E_X)$ where E_X is a C^1 -submanifold of $T_X N_A$, for each $X \in N_A$ [3]. $p_X(\cdot)$ is the projection map from TR_A^{3q} onto R_A^{3q} , and $l_{W_A}(\cdot)$ is the inclusion map, $l_{W_A}(\cdot): W_A \ni Y \rightarrow Y \in TR_A^{3q}$.

We call the set W_A , defined by the constraints imposed on the space coordinates X^1, X^2, \dots, X^{3q} and velocity coordinates ⁽²⁾ $\eta^1, \eta^2, \dots, \eta^{3q}$ of the set \mathcal{P} of particles, the extended configuration space of A . The set N_A is the configuration space of the system A .

A function $F_A(\cdot): TR_A^{3q} \ni Y \rightarrow R^{3q}$ is also given, which defines the force acting in TR_A^{3q} .

DEFINITION 1. A quartet $A = (\mathcal{P}_A, m_A, W_A, F_A(\cdot))$, where $(\mathcal{P}_A, m_A) = (\mathcal{P}, m)$, is called the lumped mechanical system observed in the fixed internal reference system $R^3 \times R_t$ and with perfectly smooth and scleronomic constraints. \square

In the case when W_A is the tangent bundle of N_A (i.e., $E_X = T_X N_A$, for each $X \in N_A$), the constraints imposed on the space and velocity coordinates of the set \mathcal{P} of particles are the holonomic constraints.

A subsystem of the system A is defined in the following way.

DEFINITION 2. By a subsystem of a lumped mechanical system $A = (\mathcal{P}_A, m_A, W_A, F_A(\cdot))$ we mean a quartet $A' = (\mathcal{P}_A, m_A, W'_A, F_A(\cdot))$ where W'_A is a subset of W_A . \square

In Definition 2 we do not demand that W'_A be endowed with the structure of a fibre bundle.

2.1. Mathematical notes

Given a C^1 -function $X(\cdot): t \rightarrow X(t) \in R_A^{3q}$ defined on an open interval $\text{Dom}(X(\cdot))$ in R_t corresponds to the function $Y(\cdot) \triangleq \tilde{D}X(\cdot): t \rightarrow \tilde{D}X(t) \in TR_A^{3q}$ given by

$$\text{Dom}(X(\cdot)) \ni t \rightarrow \tilde{D}X(t) \triangleq \left(X(t), \left(\frac{d}{dt} X \right)(t) \right),$$

where $\left(\frac{d}{dt} X \right)(t) \in T_{X(t)} R_A^{3q}$.

The function $\tilde{D}X(\cdot)$, or the corresponding parametrized curve in TR_A^{3q} , is called the lifting of $X(\cdot)$. The vector $\left(\frac{d}{dt} X \right)(t)$ is the velocity vector of the point X moving along the trajectory of the system A corresponding to the function $X(\cdot)$, at time t .

⁽¹⁾ For M being a differentiable manifold, TM denotes the tangent bundle to M , and for each $x \in M$, $T_x M$ is the tangent space to M at x (cf. [3, 4]).

⁽²⁾ We write Y for a point from the space $TR_A^{3q} \cong R_A^{3q} \times R^{3q}$, $Y = (X^1, \dots, X^{3q}, \eta^1, \dots, \eta^{3q})$. In our considerations the coordinates of a point $Y \in W_A$ are defined as the coordinates of Y in the ambient space TR_A^{3q} .

The second tangent bundle of (the manifold) R_A^{3q} is the tangent bundle $T(TR_A^{3q})$ of the bundle (the manifold) TR_A^{3q} [1–3].⁽³⁾ $T^2R_A^{3q}$ is the trivial bundle,

$$T^2R_A^{3q} \cong R_A^{3q} \times R^{3q} \times R^{3q} \times R^{3q}.$$

Let $Y(\cdot) = (X, \eta)(\cdot) : t \rightarrow TR_A^{3q}$ be a C^1 -function defined on an open interval in R_t . The function $Y(\cdot)$ corresponds to the function $\tilde{D}Y(\cdot)$ given by

$$\text{Dom}(Y(\cdot)) \ni t \rightarrow \tilde{D}Y(t) \stackrel{\Delta}{=} \left(X(t), \eta(t), \left(\frac{d}{dt} X \right)(t), \left(\frac{d}{dt} \eta \right)(t) \right),$$

where $\left(\frac{d}{dt} (X, \eta) \right)(t) \in T_{(X, \eta)(t)}^2 R_A^{3q}$. And the C^2 -function $X(\cdot)$ defined on an open interval in R_t corresponds to the function

$$\text{Dom}(X(\cdot)) \ni t \rightarrow D^2X(t) = \left(X(t), \left(\frac{d}{dt} X \right)(t), \left(\frac{d}{dt} X \right)(t), \left(\frac{d^2}{dt^2} X \right)(t) \right),$$

where $\left(\frac{d}{dt} X, \frac{d^2}{dt^2} X \right)(t) \in T_{(X, \dot{X})(t)}^2 R_A^{3q}$, $(\dot{X}) = \left(\frac{d}{dt} X \right)(t)$. The function $\tilde{D}^2X(\cdot)$, or the corresponding parametrized curve in $T^2R_A^{3q}$, is called the second lifting of $X(\cdot)$.

For Z being a point from $T^2R_A^{3q}$, the following symbols are used for the coordinates of Z :

$$Z = (X^1, \dots, X^{3q}, \eta^1, \dots, \eta^{3q}, \zeta^1, \dots, \zeta^{3q}, \xi^1, \dots, \xi^{3q}).$$

ξ^1, \dots, ξ^{3q} are the acceleration coordinates, and for $Z(\cdot) = \tilde{D}^2X(\cdot)$, $\xi(t) = \left(\frac{d^2}{dt^2} X \right)(t)$ is the acceleration vector of the point X moving along the trajectory of the system A , corresponding the function $X(\cdot)$, at time t .

As it has been assumed, the configuration space N_A is an n -dimensional C^2 -submanifold of R_A^{3q} . Let us recall that ⁽⁴⁾ $TN_A = \left\{ (X, \eta) \in R_A^{3q} \times R^{3q} : \text{there exists a } C^1\text{-function } X(\cdot) : t \rightarrow N_A \text{ defined on an open interval in } R_t \text{ containing } 0, \text{ such that } X = X(0) \text{ and } \eta = \left(\frac{d}{dt} X \right)(0) \right\}$.

The topological subspace TN_A of $TR_A^{3q} \cong R_A^{3q} \times R^{3q}$ has the structure of $2n$ -dimensional C^1 -submanifold of $R_A^{3q} \times R^{3q}$.

Let $\{(O_\alpha, g_\alpha(\cdot))\}_{\alpha \in A} : A$ being a set of indices, O_α an open subset of R_A^{3q} and $g_\alpha(\cdot) \in C^2(O_\alpha, R^{3q-n})$, be a family of constraints for N_A ; i.e., $U_\alpha \stackrel{\Delta}{=} O_\alpha \cap N_A \neq \emptyset, \bigcup_{\alpha \in A} U_\alpha = N_A$, $\text{rank } (Dg_\alpha)_{X \in U_\alpha} = 3q - n$, and $U_\alpha = \{X \in O_\alpha : g_\alpha(X) = 0\}$.⁽⁵⁾ Then the equations

⁽³⁾ In the paper we write $T^2R_A^{3q}$ for $T(TR_A^{3q})$.

⁽⁴⁾ We consider the tangent bundle to the manifold in the sense of the space of tangent bundle.

⁽⁵⁾ We use the symbol $(Dg)(\cdot)$ to denote the derivative of a map $g(\cdot)$ when the domain of $g(\cdot)$ is an open subset in Euclidean space, while we use the symbol $(dg)(\cdot)$ when the domain of $g(\cdot)$ is a general manifold. If $\text{Dom}(g(\cdot)) = R^m$ and $y = (x^{k_1}, \dots, x^{k_l}), 1 \leq k_1 < k_2 < \dots < k_l \leq m, (D_y g)(\cdot)$ denotes the derivative of $g(\cdot)$ with respect to the coordinates x^{k_1}, \dots, x^{k_l} .

$$(2.1) \quad \begin{aligned} g_\alpha(X) &= 0, \\ (Dg_\alpha)_X \cdot \eta^T &= 0, \end{aligned}$$

where $\alpha \in A: (X, \eta) \in O_\alpha \times R^{3q}$ define the constraints for TN_A in the ambient space TR_A^{3q} .

The second tangent bundle T^2N_A is defined as the following subset of $T^2R_A^{3q} \cong R_A^{3q} \times R^{3q} \times R^{3q} \times R^{3q}$:

$$T^2N_A = \left\{ (X, \eta, \zeta, \xi) \in R_A^{3q} \times R^{3q} \times R^{3q} \times R^{3q}: \text{there exists a } C^1\text{-function } Y(\cdot): t \rightarrow TN_A \text{ defined on an open interval in } R_t \text{ containing } 0, \text{ such that } (X, \eta) = Y(0) \text{ and } (\zeta, \xi) = \left(\frac{d}{dt} Y \right)(0) \right\}.$$

T^2N_A has the structure of $4n$ -dimensional (topological) submanifold of $R_A^{3q} \times R^{3q} \times R^{3q} \times R^{3q}$.

Let N_A be a C^3 -submanifold. The equations

$$(2.2) \quad \begin{aligned} g_\alpha(X) &= 0, \\ (Dg_\alpha)_X \cdot \eta^T &= 0, \\ (Dg_\alpha)_X \cdot \zeta^T &= 0, \\ (Dg_\alpha)_X \cdot \xi^T + (D^2g_\alpha)_X \cdot (\eta^T \times \eta^T) &= 0, \end{aligned}$$

where $\alpha \in A: (X, \eta, \zeta, \xi) \in O_\alpha \times R^{3q} \times R^{3q} \times R^{3q}$ define the constraints for T^2N_A in the ambient space $T^2R_A^{3q} \cong R_A^{3q} \times R^{3q} \times R^{3q} \times R^{3q}$. (The condition $\text{rank } (Dg_\alpha)_{X \in U_\alpha} = 3q - n$, for each $X \in N_A$ and $\alpha \in A: X \in U_\alpha$, ensures that Eqs. (2.2) are locally linearly independent, for each $X \in N_A$ and $\alpha \in A: X \in U_\alpha$).

Let us note that the coordinates of $Y \in TN_A$ are defined here as the coordinates of Y in the ambient space TR_A^{3q} , and the coordinates of a point Z from T^2N_A are defined as the coordinates of Z in the ambient space $T^2R_A^{3q}$. Physical (kinematical) interpretation of the coordinates of $Y \in TN_A$ (and $Z \in T^2N_A$, respectively) is the same as for the coordinates in the ambient space TR_A^{3q} (and $T^2R_A^{3q}$, respectively).

The contraction of the second tangent bundle is given by

$$\bar{T}^2N_A = \{X, \eta, \zeta, \xi\} \in T^2N_A: \quad \zeta = \eta\}.$$

For fixed $(X, \eta) \in TN_A$, the projection $p_\xi(\bar{T}^2_{(X, \eta)}N_A)$ of $\bar{T}^2_{(X, \eta)}N_A$ on the space R^{3q} of points ξ is the affine subspace of all ξ in R^{3q} satisfying

$$(2.3) \quad (Dg_\alpha)_X \cdot \xi^T = -(D^2g_\alpha)_X \cdot (\eta^T \times \eta^T),$$

where $\alpha \in A: X \in U_\alpha$. It is easy to see that $p_\xi(\bar{T}^2_{(X, \eta)}N_A)$ is the translation of the (linear) subspace $T_\eta(T_XN_A)$ of R^{3q} by the vector

$$(2.4) \quad \xi^\perp(X, \eta) = -(Dg_\alpha)_X^T \cdot [(Dg_\alpha)_X \cdot (Dg_\alpha)_X^T]^{-1} \cdot (D^2g_\alpha)_X \cdot (\eta^T \times \eta^T)$$

from the orthogonal complement of the tangent space $T_\eta(T_XN_A)$ in the space R^{3q} of vectors ξ .

Set $M_A = \text{Diag}(m_1, m_1, m_1, \dots, m_q, m_q, m_q)$. We shall see in the following that if N_A is the configuration space of a holonomic system \mathcal{A} , then $M_A \cdot \xi^\perp(X, \eta)$ is exactly the reaction force vector of the constraints $W_A = TN_A$, at the point $(X, \eta) \in TN_A$.

For a general nonholonomic system $\Lambda = (\mathcal{P}_\Lambda, m_\Lambda, \mathbf{W}_\Lambda, F_\Lambda(\cdot))$, W_Λ is a differentiable submanifold of TN_Λ , with the additional structure of a fibre bundle above N_Λ . Let $\{(O_\beta, h_\beta(\cdot))\}_{\beta \in B}$, B being a set of indices, be a family of (differentiable) constraints for W_Λ in TR_Λ^{3q} .

The tangent bundle TW_Λ is the set of points given by

$$TW_\Lambda = \left\{ (X, \eta, \zeta, \xi) \in R_\Lambda^{3q} \times R^{3q} \times R^{3q} \times R^{3q} : \text{there exists a } C^1\text{-function } Y(\cdot) = (X, \eta)(\cdot) : t \rightarrow W_\Lambda \text{ defined on an open interval in } R_t \text{ containing } 0, \text{ such that } (X, \eta) = Y(0) \text{ and } (\zeta, \xi) = \left(\frac{d}{dt} Y \right) (0) \right\}.$$

For W_Λ being a C^2 -submanifold, TW_Λ has the structure of a C^1 -submanifold of $T^2R_\Lambda^{3q}$. The (locally linearly independent) constraints defining TW_Λ are given by

$$h_\beta(X, \eta) = 0, \quad \beta \in B : (X, \eta) \in O_\beta, \\ (Dh_\beta)_{(X, \eta)} \cdot (\zeta^T, \xi^T)^T = 0, \quad (X, \eta, \zeta, \xi) \in O_\beta \times R^{3q} \times R^{3q}.$$

If $W_\Lambda = TN_\Lambda$, then $TW_\Lambda = T^2N_\Lambda$.

The contraction $\bar{T}W_\Lambda$ of TW_Λ is given by

$$\bar{T}W_\Lambda \stackrel{\Delta}{=} \{(X, \eta, \zeta, \xi) \in TW_\Lambda : \zeta = \eta\}; \\ \bar{T}_{(X, \eta)} W_\Lambda \stackrel{\Delta}{=} \{(\zeta, \xi) \in T_{(X, \eta)} W_\Lambda : \zeta = \eta\}.$$

2.2. The d'Alembert space of a lumped mechanical system

We consider a lumped mechanical system $\Lambda = (\mathcal{P}_\Lambda, m_\Lambda, \mathbf{W}_\Lambda, F_\Lambda(\cdot))$ observed in a fixed inertial reference system $R^3 \times R_t$, with perfectly smooth and scleronomic constraints.

DEFINITION 3. Let $Y(\cdot) : (X, \eta)(\cdot) : t \rightarrow Y(t) \in W_\Lambda$ be a function defined on an open and maximal interval $\text{Dom}(Y(\cdot))$ in R_t containing 0.

The function $Y(\cdot)$ is the solution of the mechanical system Λ if it is differentiable, and

i. $Y(\cdot) = \tilde{D}X(\cdot)$;

ii. for each $t \in \text{Dom}(Y(\cdot))$, there exists a (reaction force) vector $R = R(Y(t))$ from $C^1(T_\eta E_X)$ (the orthogonal complement of the tangent space $T_\eta E_X$ to E_X at $\eta \in E_X$ in the space R^{3q} of vectors ξ) such that ⁽⁶⁾

$$p_\xi(\tilde{D}Y(t)) = M_\Lambda^{-1} \cdot (F_\Lambda(Y(t)) + R(Y(t))). \quad \square$$

REMARK 1. For W_Λ being the extended configuration space of a mechanical system $\Lambda = (\mathcal{P}_\Lambda, m_\Lambda, \mathbf{W}_\Lambda, F_\Lambda(\cdot))$, we assume implicitly that: for every two points Y' and Y'' in W_Λ there exists a force field $F_{\Lambda_d}(\cdot)$ in TR_Λ^{3q} such that for some solution $Y_d(\cdot)$ of the system $\Lambda_d \stackrel{\Delta}{=} (\mathcal{P}_\Lambda, m_\Lambda, \mathbf{W}_\Lambda, F_{\Lambda_d}(\cdot))$, $Y_d(t') = Y'$ and $Y_d(t'') = Y''$, for some $t', t'' \in \text{Dom}(Y_d(\cdot))$, and $p_\xi(\tilde{D}Y_d(t)) = M_\Lambda^{-1} \cdot F_{\Lambda_d}(Y_d(t))$, for all $t \in [t', t'']$.

⁽⁶⁾ Recall that $M_\Lambda = \text{Diag}(m_1, m_1, m_1, \dots, m_q, m_q, m_q)$ and $p_\xi(\cdot)$ is the projection map from $T^2R_\Lambda^{3q}$ on the space R^{3q} of vectors ξ .

In other words, there is such a force field $F_{A_A}(\cdot)$ which controls the state change of A from Y' to Y'' , with zero reaction force along the corresponding trajectory from Y' to Y'' . \square

DEFINITION 4. *The dynamic space \mathcal{M}_A of a mechanical system A (the space of motion of A) is the following subset of the extended configuration space of A :*

$$\mathcal{M}_A = \{Y \in W_A: \text{there exists a solution } Y(\cdot) \text{ of } A, \text{ such that } Y = Y(0)\}. \quad \square$$

Write \mathcal{R}_A for the set of solutions of A .

We consider the class of mechanical systems $A = (\mathcal{P}_A, m_A, W_A, F_A(\cdot))$ such that for every system A the dynamic space \mathcal{M}_A of A is a differentiable submanifold of W_A (of TR_A^{3g}), and the motion of the system (evolution of the state in time) is the flow defined by a vector field (\cdot) on \mathcal{M}_A [4].

DEFINITION 5. *Let the dynamic space \mathcal{M}_A of a lumped mechanical system $A = (\mathcal{P}_A, m_A, W_A, F_A(\cdot))$ be a differentiable submanifold of W_A , and let $f(\cdot)$ be a vector field on \mathcal{M}_A . Assume that $f(\cdot)$ defines a flow $\sigma(\cdot)$ on \mathcal{M}_A .*

We say that $f_A(\cdot) \stackrel{\Delta}{=} f(\cdot)$ is the vector field generated by the system A on its dynamic space \mathcal{M}_A if

$$\bigcup_{Y \in \mathcal{M}_A} \sigma(Y, \cdot) = \mathcal{R}_A. \quad \square$$

The given vector field $\chi(\cdot)$ on a differentiable manifold M corresponds to the subset

$$\tilde{\chi} = \text{Im}(\chi(\cdot))$$

in TM (the image of the map $\chi(\cdot)$). If $\chi(\cdot)$ is the C^r -vector field on the C^k -manifold M , $0 \leq r \leq k-1$, then $\tilde{\chi}$ has the structure of the C^r -submanifold of TM . Using this observation we obtain an equivalent version of the definition of the vector field, which is useful in the following considerations.

DEFINITION 6. *A vector field on the differentiable C^k -manifold M is a subset $\tilde{\chi}$ of TM , such that $(p_x \circ l_{\tilde{\chi}})(\cdot)$, where $l_{\tilde{\chi}}(\cdot): \tilde{\chi} \ni (x, \zeta) \rightarrow (x, \zeta) \in TM$, is the one-to-one map of $\tilde{\chi}$ onto M .*

If in addition $\tilde{\chi}$ has the structure of the C^r -submanifold of TM , $0 \leq r \leq k-1$, then $\tilde{\chi}$ is the C^r -vector field. \square

(\cdot) Let M be a differentiable C^k -manifold, and let ζ denote the vector tangent to M at some point $x \in M$. We recall that a vector field on the manifold M is a map $\chi(\cdot)$ of M into the tangent bundle (the manifold) TM such that

$$\chi(x) = (x, p_\zeta(\chi(x))),$$

for each $x \in M$ where $p_\zeta(\cdot)$ is the projection map, $p_\zeta(\cdot): TM \ni (x, \zeta) \rightarrow p_\zeta(x, \zeta) = \zeta \in T_x M$ [3, 4].

If, in addition, $\chi(\cdot)$ is the C^r -map, $0 \leq r \leq k-1$, then $\chi(\cdot)$ is the C^r -vector field.

A flow on M is, by assumption, a map $\sigma(\cdot): U \subseteq M \times \mathbb{R}_t \rightarrow M$, where U is an open subset containing $M \times \{0\}$, such that: $\sigma(x, 0) = x$, and $\sigma(x, t' + t'') = \sigma(\sigma(x, t'), t'')$ whenever both sides of the equation are defined. It can be proved that every differentiable vector field on a differentiable manifold M defines a flow on M .

The vector field $f_A(\cdot)$ generated by a mechanical system A corresponds to the subset

$$\tilde{f}_A = \bigcup_{Y \in \mathcal{M}_A} f_A(Y)$$

in TW_A (in $T\mathcal{M}_A$). In view of Definition 6, \tilde{f}_A is also called the vector field generated by the system A on its dynamic space \mathcal{M}_A .

REMARK 2. It follows from the definition of the solution and the definition of the vector field generated by a lumped mechanical system A that

$$p_\zeta(f_A(X, \eta)) = \eta,$$

for each $(X, \eta) \in \mathcal{M}_A$ and

$$(2.5) \quad R = M_A \cdot p_\xi(f_A(Y)) - F_A(Y),$$

for each $Y \in \mathcal{M}_A$.

From Eq. (2.5) we obtain that for the system A which generates a vector field on its dynamic space \mathcal{M}_A , the reaction force vector R remains unchanged, independently of the choice of the solution $Y(\cdot)$ of A passing through a given point $Y \in \mathcal{M}_A$. \square

For fixed $(X, \eta) \in W_A$, $p_{||}(\cdot)$ is the projection map from R^{3q} on $T_\eta E_X$ (the tangent space to the submanifold E_X of $T_X N$, at the point $\eta \in E_X$), and $p_\perp(\cdot)$ is the projection map from R^{3q} on $C^\perp(T_\eta E_X)$ (the orthogonal complement of $T_\eta E_X$ in R^{3q}). For $\xi \in R^{3q}$, $p_{||}(\xi)$ and $p_\perp(\xi)$ are considered here as vectors in R^{3q} .

The following definition of the d'Alembert space of a lumped mechanical system A is of basic importance in our considerations concerning the dynamic spaces and the dynamic equations of lumped mechanical systems.

DEFINITION 7. The d'Alembert space S_A of a lumped mechanical system $A = (\mathcal{P}_A, m_A, W_A, F_A(\cdot))$ is defined as the following subset of TW_A (and hence, of $T^2 R_A^{3q}$):

$$S_A = \{(X, \eta, \zeta, \xi) \in TW_A: \zeta = \eta \quad \text{and} \quad p_{||}(M_A \cdot \xi) = p_{||}(F_A(X, \eta))\}. \quad \square$$

REMARK 3. Note that for a vector $\xi \in R^{3q}$ there is a vector $R \in C^\perp(T_\eta E_X)$ such that $\xi = M_A^{-1} \cdot (F_A(X, \eta) + R)$ if, and only if,

$$p_{||}(M_A \cdot \xi) = p_{||}(F_A(X, \eta)).$$

Thus the conditions i. and ii. in Definition 3 are equivalent to

$$(2.6) \quad \tilde{D}Y(t) \in S_A,$$

for all $t \in \text{Dom}(Y(\cdot))$. \square

REMARK 4. Let us consider the case when the dynamic space \mathcal{M}_A of a lumped mechanical system $A = (\mathcal{P}_A, m_A, W_A, F_A(\cdot))$ is a differentiable submanifold of W_A , and \mathcal{M}_A has the structure of a fibre bundle $(\mathcal{M}_A, p_X \circ l_{\mathcal{M}_A}(\cdot), p_X(\mathcal{M}_A))$ where: $p_X(\mathcal{M}_A)$ is a C^2 -submanifold of N_A , $\mathcal{M}_A = \bigcup_{X \in p_X(\mathcal{M}_A)} (X, E'_X)$, and E'_X is a C^1 -submanifold of $T_X N_A$ with Dim

$E'_X < \text{Dim } E_X$. Then it follows from Definition 3 of the solution of a lumped mechanical system that for each $(X, \eta) \in \mathcal{M}_A$, the available reaction force vector range remains bounded to $C^\perp(T_\eta E_X)$.

From the above it follows that if $\mathcal{A}' = (\mathcal{P}_A, m_A, W'_A, F_A(\cdot))$ is a subsystem of the system $\mathcal{A} = (\mathcal{P}_A, m_A, W_A, F_A(\cdot))$, one would take S_A — the d'Alembert space of \mathcal{A} , as the d'Alembert space of the subsystem \mathcal{A}' . \square

3. The dynamic spaces and the equations of motion of lumped mechanical systems

In this Section we analyse the dynamic spaces and vector fields generated by lumped mechanical systems. A general procedure for finding the space of motion and the dynamic equation of a mechanical system is proposed.

The main conclusion, which is also valid for general dynamic systems, is the following. The space of motion of a mechanical system \mathcal{A} is the set union of all invariant submanifolds of the extended configuration space of \mathcal{A} . We prove this in the Appendix.

THEOREM 1. *Let $\mathcal{A} = (\mathcal{P}_A, m_A, W_A, F_A(\cdot))$ be a lumped mechanical system which has the dynamic space \mathcal{M}_A being a C^1 -submanifold of W_A , and which generates a vector field $f_A(\cdot)$ on \mathcal{M}_A .*

Then $(^8)$

$$\tilde{f}_A \subseteq S_A.$$

Proof. Let $Y = (X, \eta) \in \mathcal{M}_A$. Set $Z = f_A(X, \eta)$. It suffices to prove that $Z \in S_A$.

But $Z = \tilde{D}Y(t)$, for some solution $Y(\cdot)$ of \mathcal{A} , at some $t \in \text{Dom}(Y(\cdot))$, and $\tilde{D}Y(t) \in S_A$, for all $t \in \text{Dom}(Y(\cdot))$. \square

The regular systems are the most often considered class of lumped mechanical systems. We except that apart from the systems constructed as systems which exhibit some kind of discontinuous modes, the well-posed mechanical systems are regular.

DEFINITION 8. *A lumped mechanical system $\mathcal{A} = (\mathcal{P}_A, m_A, W_A, F_A(\cdot))$ is said to be regular if $\mathcal{M}_A = W_A$ and if it generates a vector field on W_A . \square*

Using Theorem 1, we get in conclusion the following theorem.

THEOREM 2. *Let $\mathcal{A} = (\mathcal{P}_A, m_A, W_A, F_A(\cdot))$ be a regular mechanical system.*

If the d'Alembert space S_A of \mathcal{A} is the vector field on W_A (in the sense of Definition 6), then

$$\tilde{f}_A = S_A. \quad \square$$

For nonregular lumped mechanical systems, we have the following theorem which proposes a general procedure for finding the space of motion and the dynamic equation of the system. We illustrate this procedure in the Example, Sect. 4 of the paper, where a nonregular mechanical system is being analysed.

THEOREM 3. *Let the projection $p_Y(S_A)$ of the d'Alembert space S_A of a lumped mechanical system $\mathcal{A} = (\mathcal{P}_A, m_A, W_A, F_A(\cdot))$ on the position-velocity space TR_A^{3q} be a C^1 -sub-*

⁽⁸⁾ Let us recall that $\tilde{f}_A = \text{Im}(f_A(\cdot)) \subseteq TW_A$.

manifold of W_A , and let S_A be a C^0 -vector field on $p_Y(S_A)$:

$$S_A = \bigcup_{Y \in p_Y(S_A)} (Y, \bar{f}_A(Y)),$$

where $\bar{f}_A(\cdot): p_Y(S_A) \ni Y \rightarrow \bar{f}_A(Y) \in T_Y(p_Y(S_A))$ is a C^0 -map.

If the differential equation $\frac{dY}{dt} = \bar{f}_A(Y)$ on $p_Y(S_A)$ has uniquely defined solutions, then

$$\mathcal{M}_A = p_Y(S_A),$$

the system A generates a vector field $f_A(\cdot)$ on \mathcal{M}_A , and

$$\tilde{f}_A = S_A.$$

P r o o f. Note that $p_Y(S_A) \subseteq W_A$. Since each solution $Y(\cdot) = Y(Y_0, \cdot)$ of the differential equation $\frac{dY}{dt} = \bar{f}_A(Y)$ (defined on $p_Y(S_A)$) satisfies the relation (2.6), for each $t \in \text{Dom}(Y(\cdot))$, then each of them is a part of some solution of the system A .

It follows then that $p_Y(S_A) \subseteq \mathcal{M}_A$.

We shall now prove that $\mathcal{M}_A \subseteq p_Y(S_A)$.

Fix a point $Y^0 \in \mathcal{M}_A \subseteq W_A$. It follows from the definition of the dynamic space of the mechanical system A that there exists a solution $Y(\cdot)$ of A passing through Y^0 at $t = 0$ and, by assumption, the function $Y(\cdot)$ is differentiable. Thus we have a well-defined vector $(\zeta^0, \xi^0) \triangleq \left(\frac{d}{dt} Y \right) (0)$ which is tangent to W_A at the point Y^0 , and by the definition of the solution of a mechanical system A , $(Y^0, \zeta^0, \xi^0) \in S_A$.

It follows then that $Y^0 \in p_Y(S_A)$ and hence $\mathcal{M}_A \subseteq p_Y(S_A)$. \square

In the Appendix we prove Theorem A.6 which is the extension of Theorem 3. It encloses a multi-step procedure which enables to exclude these points in W_A which are not in the space of motion of the system A . Theorem 3 concerns only the case when $\mathcal{M}_A = p_Y(S_A)$ and $\tilde{f}_A = S_A$.

REMARK 5. Theorem 3 has the following extension which is useful in the considerations in Sect. 4 where the example of the nonregular lumped mechanical system is analysed.

Let the projection $p_Y(S_A)$ of the d'Alembert space S_A of a lumped mechanical system A on the position-velocity space of A have the structure of the set union $\bigcup_{\gamma \in \Gamma} W^{(\gamma)}$ of disjoint C^1 -submanifolds $W^{(\gamma)}$ of W_A , Γ — a set of indices, such that S_A when restricted to $W^{(\gamma)}$ is a C^0 -vector field on $W^{(\gamma)}$, for each $\gamma \in \Gamma$.

In such a case, if for each $\gamma \in \Gamma$ the corresponding S_A differential equation on $W^{(\gamma)}$ has uniquely defined solutions, then

$$\mathcal{M}_A = p_Y(S_A),$$

the system A generates a vector field $f_A(\cdot)$ on \mathcal{M}_A , and

$$\tilde{f}_A = S_A. \quad \square$$

4. Regularity criteria for lumped mechanical systems

Regularity of a mechanical system is an important property of the system. Except for the lumped mechanical systems constructed as systems which exhibit some kind of discontinuous modes, the well-posed mechanical systems are regular (Definition 8). In this chapter the conditions ensuring regularity of a lumped mechanical system are proposed.

We make the following standing assumptions for the extended configuration space W_A of a mechanical system A and for the constraints defining W_A .

I. W_A is a C^2 -submanifold of TR_A^{3q} .

II. W_A has the structure of a fibre bundle $W_A = (W_A, p_X \circ l_{W_A}(\cdot), N_A)$, where N_A is an n -dimensional C^3 -submanifold of R_A^{3q} , $W_A = \bigcup_{X \in N_A} (X, E_X)$, and for each $X \in N_A$,

E_X is a C^2 -submanifold of $T_X N_A$.

From I and II it follows that $\text{Dim } E_X = n - m$ for some number $0 \leq m \leq n$, which is constant for all $X \in N_A$.

III. There exists a family $\{(O_\beta, g_\beta(\cdot))\}_{\beta \in B}$, B being a set of indices, of C^2 -constraints for W_A , such that for each $\beta \in B$ and $(X, \eta) \in O_\beta \cap W_A$, $\text{rank } (D_\eta g_\beta)_{(X, \eta)} = 3q - n + m$.

IV. $F_A(\cdot) \in C^1(TR_A^{3q}, R^{3q})$.

THEOREM 4. *Assumptions I-IV ensure that the lumped mechanical system A is regular.*

P r o o f. At the first step, we shall prove that the equation

$$(4.1) \quad p_{||}(M_A \cdot \xi) = p_{||}(F_A(X, \eta)) \quad \text{in } \bar{T}W_A$$

has a unique solution with respect to ξ , for each $(X, \eta) \in W_A$, and that the solution is a differentiable function of $(X, \eta) \in W_A$. And next we shall use Theorem 3 to obtain the thesis (the system A generates a C^1 -vector field on $\mathcal{M}_A = W_A$).

From Assumption III it follows that for each point $Y \in W_A$ there exists an open neighbourhood O_Y of Y in TR_A^{3q} and a map $h_Y(\cdot): O_Y \rightarrow R^{2(3q-n)+m}$ such that $h_Y(\cdot)$ defines C^2 -constraints for $W_A \cap O_Y$, and there is such a subsequence $(\bar{h}_Y^1, \dots, \bar{h}_Y^{3q-n+m})(\cdot)$ of $(h_Y^1, \dots, h_Y^{(3q-n)+m})(\cdot)$, that for each $X^0 \in p_X(O_Y)$ $\bar{h}_Y(X^0, \cdot)$ defines C^2 -constraints for E_{X^0} in

$$p_\eta(\{(X, \eta) \in O_Y: X = X^0\}).$$

From the family $\{(O_Y, h_Y(\cdot))\}_{Y \in W_A}$ of the constraints defined above, one can extract a countable subfamily which we denote by $\{(O_\gamma, h_\gamma(\cdot))\}_{\gamma \in \Gamma}$, Γ — being a set of indices.

For each $(X^0, \eta^0) \in O_\gamma$, $\bar{h}_\gamma(X^0, \cdot)$ defines the constraints for E_{X^0} in $p_\eta(\{(X, \eta) \in O_\gamma: X = X^0\})$.

In what follows, the symbols O_γ , $h_\gamma(\cdot)$ and $\bar{h}_\gamma(\cdot)$ have the meaning defined above.

The locally linearly independent constraints for the contraction $\bar{T}W_A = \{(X, \eta, \zeta, \xi) \in TW_A: \zeta = \eta\}$ of the tangent bundle TW_A are given by

$$h_\gamma(X, \eta) = 0,$$

$$\zeta - \eta = 0,$$

$$(D_\eta \bar{h}_\gamma)_{(X, \eta)} \cdot \xi^T + (D_X \bar{h}_\gamma)_{(X, \eta)} \cdot \eta^T = 0, \quad \gamma \in \Gamma: (X, \eta) \in O_\gamma, \quad (X, \eta, \zeta, \xi) \in O_\gamma \times R^{3q} \times R^{3q}.$$

For fixed $(X, \eta) \in W_A$, $p_\xi(\bar{T}_{(X, \eta)}W_A)$ is an $(n-m)$ -dimensional affine subspace of R^{3q} given by

$$(4.2) \quad (D_\eta \bar{h}_\gamma)_{(X, \eta)} \cdot \xi^T = -(D_X \bar{h}_\gamma)_{(X, \eta)} \cdot \eta^T,$$

where $\gamma \in \Gamma: (X, \eta) \in O_\gamma \cdot p_\xi(\bar{T}_{(X, \eta)}W_A)$ is the translation of the (linear) subspace $T_\eta E_X$ of R^{3q} by (uniquely defined) vector $\xi^\perp(X, \eta)$ from the orthogonal complement $C^\perp(T_\eta E_X)$ of $T_\eta E_X$ in R^{3q} ; the vector $\xi^\perp(X, \eta)$ is given by the expression ⁽⁹⁾:

$$(4.3) \quad \xi^\perp(X, \eta) = -(D_\eta \bar{h}_\gamma)_{(X, \eta)}^T \cdot [(D_\eta \bar{h}_\gamma)_{(X, \eta)} \cdot (D_\eta \bar{h}_\gamma)_{(X, \eta)}^T]^{-1} \cdot (D_X \bar{h}_\gamma)_{(X, \eta)} \cdot \eta^T$$

$$(\gamma \in \Gamma: (X, \eta) \in O_\gamma).$$

Equation (4.1) (in $\bar{T}W_A$) is equivalent to

$$(4.4) \quad p_{||}(M_A \cdot \xi_{||}) = -p_{||}(M_A \cdot \xi_\perp) + p_{||}(F_A(X, \eta)),$$

where $\xi_{||}$ is the tangent part of ξ , $\xi_{||} = p_{||}(\xi)$, and ξ_\perp is the orthogonal part of ξ ; $\xi_\perp = p_\perp(\xi)$, and $\xi_\perp = \xi^\perp(X, \eta)$ is given by the expression (4.3).

For each $Y = (X, \eta) \in W_A$ set

$$(4.5) \quad \psi_A(Y) = -p_{||}(M_A \cdot \xi^\perp(Y)) + p_{||}(F_A(Y)),$$

and write Eq. (4.4) as

$$(4.6) \quad p_{||}(M_A \cdot \xi_{||}) = \psi_A(Y).$$

A point $Z = (X, \eta, \zeta, \xi)$ belongs to the d'Alembert space of the system A if, and only if, $Z \in \bar{T}W_A$ and Eq. (4.6) is satisfied for $\xi_{||} := p_{||}(\xi)$.

Equation (4.6) has a unique solution at each $Y \in W_A$ if, and only if, the d'Alembert space S_A of A is the vector field on W_A (S_A is then the C^1 -vector field generated by the system A on its dynamic space $\mathcal{M}_A = W_A$ — Theorem 3).

We shall prove now that there is a unique solution to Eq. (4.6), at each $Y \in W_A$.

For this let us first observe that

$$(4.7) \quad p_{||}(M_A \cdot T_\eta E_X) = T_\eta E_X,$$

for each $(X, \eta) \in W_A$.

In fact, suppose that Eq. (4.7) is not true for some $(X^0, \eta^0) \in W_A$. Then $p_{||}(M_A \cdot T_{\eta^0} E_{X^0})$ is a linear subspace of $T_{\eta^0} E_{X^0}$ with $\text{Dim}(p_{||}(M_A \cdot T_{\eta^0} E_{X^0})) < \text{Dim}(T_{\eta^0} E_{X^0})$. There exists then a vector $\xi' \neq 0$ in $M_A \cdot T_{\eta^0} E_{X^0}$ such that

$$(4.8) \quad p_{||}(\xi') = 0.$$

But $\xi' = M_A \cdot \xi$, for some $\xi \in T_{\eta^0} E_{X^0}$ and $\xi \neq 0$, and hence Eq. (4.8) implies

$$p_{||}(M_A \cdot \xi) = 0,$$

for some $\xi \in T_{\eta^0} E_{X^0}$ and $\xi \neq 0$ (let us recall that ξ' and ξ are considered as vectors

⁽⁹⁾ The expressions (4.2) and (4.3) are the extension of Eqs (2.3) and (2.4) for the case when the extended configuration space is a general fibre bundle, i.e., the constraints are nonholonomic.

Assumption III is in fact necessary for $\xi^\perp(X, \eta)$ to be well defined.

Let us also note that the condition $(Dh_\gamma)_{(X, \eta)} \cdot (\eta^T, \xi^T)^T = 0$ may be written equivalently as

$$\xi \in p_\xi(\bar{T}_{(X, \eta)}W_A).$$

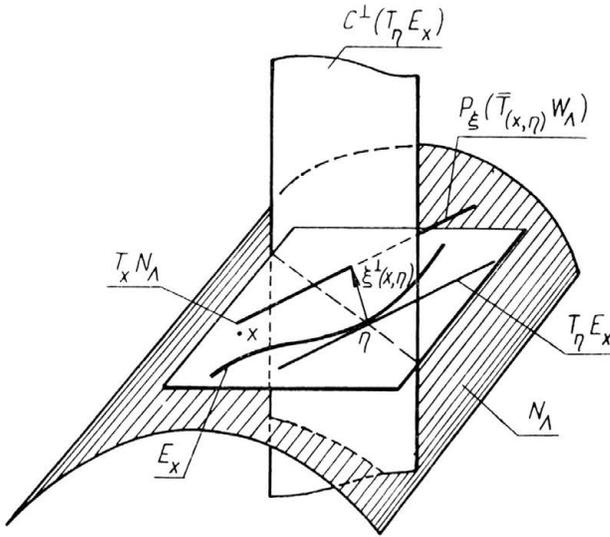


FIG. 1. Geometric objects which are explored in the description of mechanical systems. Here the spaces $p_\xi(\bar{T}(x, \eta) W_\Lambda)$, $T_\eta E_x$ and $C^\perp(T_\eta E_x)$ and the vector $\xi^\perp(x, \eta)$ have been shifted to the tangent space $T_x R_A^{3q}$ to make the illustration clear.

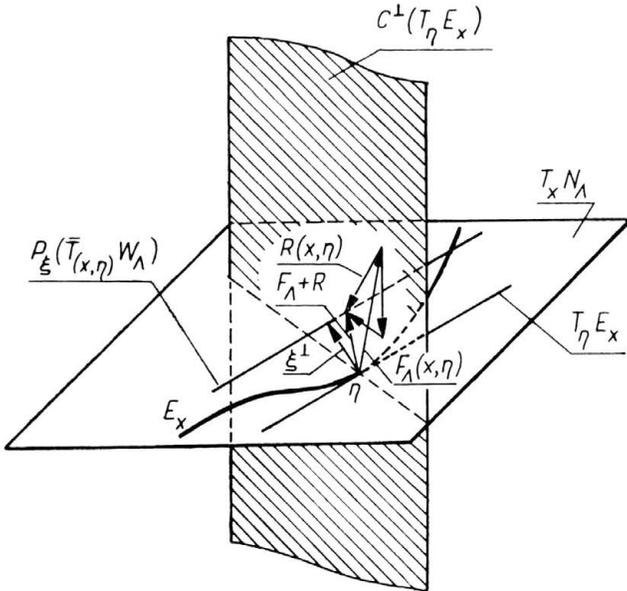


FIG. 2. A point $Y = (X, \eta)$ from the extended configuration space W_Λ belongs to the dynamic space of the system Λ if there is such a vector $R = R(Y) \in C^\perp(T_\eta E_x)$ that $F_\Lambda(Y) + R(Y) \in p_\xi(\bar{T}(x, \eta) W_\Lambda)$. For simplicity, it has been assumed that $m_1 = m_2 = \dots = m_q = 1$.

from R^{3q}) which yields $\xi \cdot M_A \cdot \xi = 0$. This, however, is not possible because M_A is the strictly positively defined matrix.

From Eq. (4.7) we obtain in conclusion that Eq. (4.6) has a solution with respect to $\xi_{||}$, at each $Y \in W_A$.

Suppose now that Eq. (4.6) has two solutions $\xi'_{||}$ and $\xi''_{||}$, for some $Y \in W_A$. Then

$$p_{||}(M_A \cdot (\xi''_{||} - \xi'_{||})) = 0$$

and $\xi''_{||} - \xi'_{||} \neq 0$ which is not possible.

Finally, then, Eq. (4.6) has a unique solution at each $Y \in W_A$, and the thesis of the theorem is proved. \square

Let us now derive an explicit expression for the acceleration vector $\xi = \xi(Y)$ of the system A . We assume that the extended configuration space W_A of A fulfils Assumptions I-III and $F_A(\cdot)$ is a C^1 -function on TR_A^{3q} .

For each $Y = (X, \eta) \in W_A$ and $\gamma \in \Gamma: Y \in O_\gamma$, $(H_\gamma)_{(X, \eta)}$ is defined as a $(3q - n + m) \times 3q$ -dimensional matrix given by

$$(4.9) \quad (H_\gamma)_{(X, \eta)} = \text{Diag}(\|(D_\eta \bar{h}_\gamma)_{(X, \eta)}\|^{-1}, \dots, \|(D_\eta \bar{h}_\gamma^{3q-n+m})_{(X, \eta)}\|^{-1}) \cdot (D_\eta \bar{h}_\gamma)_{(X, \eta)};$$

the rows of $(H_\gamma)_{(X, \eta)}$ are versors in the subspace $C^\perp(T_\eta E_X)$ of R^{3q} , and $\text{rank } (H_\gamma)_{(X, \eta)} = 3q - n + m$.

The matrix $(G_\gamma)_{(X, \eta)}$ is the Gram's matrix of $(H_\gamma)_{(X, \eta)}^T$,

$$(4.10) \quad (G_\gamma)_{(X, \eta)} = (H_\gamma)_{(X, \eta)}^T \cdot (H_\gamma)_{(X, \eta)};$$

$\text{rank } (G_\gamma)_{(X, \eta)} = 3q - n + m$.

And next, define

$$(4.11) \quad (B_\gamma)_{(X, \eta)} \triangleq \begin{bmatrix} [I - (G_\gamma)_{(X, \eta)}] \cdot M_A \\ \dots\dots\dots \\ (D_\eta \bar{h}_\gamma)_{(X, \eta)} \end{bmatrix}$$

for each $Y = (X, \eta) \in W_A$ and $\gamma \in \Gamma: Y \in O_\gamma$, and

$$(4.12) \quad \tilde{\psi}_A(Y) = \begin{bmatrix} \psi_A(Y) \\ \dots\dots \\ 0 \end{bmatrix} \uparrow 3q - n + m$$

for each $Y \in W_A$.

Equation (4.6) in $\bar{T}W_A$ is equivalent to the following system of equations:

$$(4.13) \quad \text{and} \quad \left. \begin{aligned} [I - (G_\gamma)_{(X, \eta)}] \cdot M_A \cdot \xi &= \psi_A(Y), \\ (D_\eta \bar{h}_\gamma)_{(X, \eta)} \cdot \xi &= 0, \end{aligned} \right\}$$

with respect to ξ in R^{3q} . The assumptions ensure that the system (4.13) has a unique solution at each $Y = (X, \eta) \in W_A$ (see the proof of Theorem 4), and for fixed $Y \in W_A$, the solutions ξ of the system (4.13) and $\xi_{||}$ of Eq. (4.6) are related by

$$\xi(Y) = \xi_{||}(Y) + \xi^\perp(Y),$$

where the orthogonal part $\xi^\perp(Y)$ of $\xi(Y)$ is given by the expression (4.3).

Let us write (4.13) in a more compact form as

$$(4.14) \quad (B_\gamma)_{(X, \eta)} \cdot \xi = \tilde{\psi}_A(Y), \quad \xi \in R^{3q},$$

where $\gamma \in \Gamma: Y = (X, \eta) \in O_\gamma$, and observe that by the proved existence of a unique solution of Eq. (4.6), and hence by the existence of a unique solution of Eq. (4.14),

$$\text{rank}(B_\gamma)_{(X, \eta)} = \text{rank}([(B_\gamma)_{(X, \eta)}] \tilde{\psi}_A(Y)) = 3q$$

for each $Y = (X, \eta) \in W_A$ and $\gamma \in \Gamma: Y \in O_\gamma$.

We summarize the observations in the following Corollary.

COROLLARY. For a lumped mechanical system A satisfying the conditions specified in Assumptions I-IV, the acceleration vector $\xi(Y) = \bar{f}_A(Y), \bar{f}_A(\cdot) := p_\xi \circ f_A(\cdot)$ is given by the following explicit expression:

$$\bar{f}_A(X, \eta) = \xi_{||}(X, \eta) + \xi_\perp(X, \eta),$$

where $\xi_\perp(X, \eta)$ is the vector $\xi^\perp(X, \eta)$ given in the expression (4.3), and the tangent part $\xi_{||}(X, \eta)$ is given by the solution of Eq. (4.14),

$$(4.15) \quad \xi_{||}(X, \eta) = [(B_\gamma)_{(X, \eta)}^T \cdot (B_\gamma)_{(X, \eta)}]^{-1} \cdot M_A \cdot \psi_A(Y).$$

We observe that the orthogonal part $\xi_\perp(Y)$ of $f_A(Y)$ is prescribed entirely by the constraints imposed on the space and velocity coordinates of the system.

REMARK 6. Assume for a lumped mechanical system A satisfying I-IV, that

$$M_A^{-1} \cdot \psi_A(Y) \in T_\eta E_X$$

for each $Y = (X, \eta) \in W_A$. Then the vector

$$\xi(Y) := M_A^{-1} \cdot \psi_A(Y)$$

is the (unique) solution of Eq. (4.14) (the expression (4.15) for the tangent part of the acceleration vector reduces to $\xi_{||}(Y) = M_A^{-1} \cdot \psi_A(Y)$). \square

REMARK 7. Assume for a lumped mechanical system A satisfying I-IV that

$$M_A \cdot \xi^\perp(Y) \in C^\perp(T_\eta E_X),$$

for each $Y \in W_A$. Then

$$\psi_A(Y) = p_{||}(F_A(Y)),$$

and

$$\xi_{||}(Y) = M_A^{-1} \cdot p_{||}(F_A(Y)).$$

In this case the orthogonal part of the acceleration vector does not enter the expression for $\xi_{||}(Y)$. \square

It has been assumed implicitly in the description of nonlinear lumped mechanical systems presented in this paper that the constraints are able to create a reaction force which has an arbitrarily large value (the only constraint for the reaction force vector is to be situated in $C^\perp(T_\eta E_X)$). Thus, taking into account the results in Theorem 4, we obtain in conclusion that the constraints defining the extended configuration space of a nonregular lumped mechanical system must fail the condition in Assumption III.

In other words, the d'Alembert space S_A of the system A causes shrinkage of the extended configuration space W_A to the proper subset of W_A — the dynamic space \mathcal{M}_A of A , in the case when the contraction operation applied to the tangent bundle TW_A

yields

$p_Y(\overline{TW}_A)$ is the proper subset of W_A .

Let A be a lumped mechanical system such that Assumptions I, II and IV are satisfied. Let $\{(O_\beta, g_\beta(\cdot))\}_{\beta \in B}$ be a family of C^2 -constraints for W_A . The equations defining the C^1 -constraints for \overline{TW}_A are given by

$$(4.16) \quad \begin{aligned} g_\beta(X, \eta) &= 0, \\ \zeta - \eta &= 0, \\ (D_\eta g_\beta)_{(X, \eta)} \cdot \xi^T + (D_X g_\beta)_{(X, \eta)} \cdot \zeta^T &= 0, \end{aligned}$$

where $\beta \in B: (X, \eta) \in O_\beta, (X, \eta, \zeta, \xi) \in O_\beta \times R^{3q} \times R^{3q}$ (let us note that the constraints (4.16) are not necessarily locally linearly independent).

The system

$$(4.17) \quad (D_\eta g_\beta)_{(X, \eta)} \cdot \xi^T = -(D_X g_\beta)_{(X, \eta)} \cdot \eta^T$$

of $2(3q-n)+m$ equations does not have the solution with respect to ξ at the point $(X, \eta) \in W_A$ if

$$(4.18) \quad \text{rank}[(D_\eta g_\beta)_{(X, \eta)}] < \text{rank}[(D_\eta g_\beta)_{(X, \eta)} \vdots -(D_X g_\beta)_{(X, \eta)} \cdot \eta^T].$$

These points (X, η) in W_A where the relation (4.18) holds are not in $p_Y(\overline{TW}_A)$, and hence they are not in the space of motion of the system A .

For a holonomic system A , with the family $\{(O_\alpha, g_\alpha(\cdot))\}_{\alpha \in A}$ of C^3 -constraints for N_A , the system (4.17) of equations reduces to

$$(4.19) \quad (Dg_\alpha)_X \cdot \xi^T = -(D^2 g_\alpha)_X \cdot (\eta^T \times \eta^T),$$

where $\alpha \in A: X \in O_\alpha$ and $\eta \in T_X N_A$. By noting that the system (4.19) has a solution with respect to ξ at each $(X, \eta) \in W_A$, we obtain in conclusion that the smooth holonomic lumped systems are necessarily the regular systems.

In the following example, a nonholonomic nonregular mechanical system is discussed.

EXAMPLE

Let us consider a system composed of two particles \mathcal{P}_1 and \mathcal{P}_2 having the inert masses m_1 and m_2 , respectively. To simplify notations, we assume that both the motion of \mathcal{P}_1 and \mathcal{P}_2 are one-dimensional.

Let the space coordinates x_1 and x_2 of \mathcal{P}_1 and \mathcal{P}_2 , respectively, and the velocity coordinates η_1 and η_2 of \mathcal{P}_1 and \mathcal{P}_2 , respectively, be constrained by the equation

$$(4.20) \quad (\eta_1 + \eta_2 - a)^3 - x_1 = 0.$$

Equation (4.20) defines the extended configuration space W_A of the system A . In view of Eq. (4.20), W_A has the structure of a fibre bundle $\mathbb{W}_A = (W_A, p_X \cdot l_{W_A}(\cdot), N_A)$ when N_A is the entire position space R_A^2 of A , $W_A = \bigcup_{X \in R_A^2} (X, E_X)$, and E_X is a one-dimensional affine subspace of $T_X R_A^2$ given by

$$(4.21) \quad \eta_1 + \eta_2 = \sqrt[3]{x_1} + a.$$

We assume that the system A is autonomous, i.e., $F_A(\cdot) \equiv 0$.

The d'Alembert space of the system \mathcal{A} is given by

$$S_A = \{(X, \eta, \eta, \xi) \in \overline{TW}_A : p_{||}(M_A \cdot \xi) = 0\},$$

and it decomposes into two (disjoint) subsets S'_A and S''_A :

$$S'_A = \overline{TW}'_{(1)}$$

(the condition $p_{||}(M_A \cdot \xi) = 0$ yields $\xi = 0$, at each $(X, \eta) \in W'_{(1)}$, which is identical to the condition on ξ following from the constraint relations for $\overline{TW}'_{(1)}$), and

$$S''_A = \{(X, \eta, \eta, \xi) \in \overline{TW}_{(1)} : m_1 \cdot \xi_1 - m_2 \cdot \xi_2 = 0\}.$$

We then have that S'_A is the vector field on $W'_{(1)} = p_Y(S'_A)$ given by

$$(4.23) \quad W'_{(1)} \ni Y = (0, x_2, 0, a) \rightarrow \underbrace{[0, a, 0, 0]^T}_{\xi} \in T_Y W'_{(1)},$$

and S''_A is the vector field on $W''_{(1)} = p_Y(S''_A)$ given by

$$(4.24) \quad W''_{(1)} \ni Y = (x_1, x_2, \eta_1, \eta_2) \rightarrow \left[\eta_1, \eta_2, \frac{m_2}{3(m_1 + m_2) \sqrt[3]{(x_1)^2}} \cdot \eta_1, \frac{m_1}{3(m_1 + m_2) \sqrt[3]{(x_1)^2}} \cdot \eta_1 \right]^T \in T_Y W''_{(1)}.$$

Using Theorem 3 (and also Remark 5), we obtain in conclusion that

$$\mathcal{M}_A = W_{(1)}$$

and S_A is the vector field generated by the system \mathcal{A} .

From Eq. (4.24), or using the expression (4.3), one has

$$\xi^\perp(X, \eta) = \frac{1}{6 \sqrt[3]{(x_1)^2}} \cdot \begin{bmatrix} \eta_1 \\ \dots \\ \eta_1 \end{bmatrix} \quad \text{for } (X, \eta) \in W''_{(1)},$$

and next, with the aid of Eq. (4.15) or directly from Eq. (4.24),

$$\xi_{||}(X, \eta) = \frac{\eta_1}{6 \sqrt[3]{(x_1)^2} (m_1 + m_2)} \cdot \begin{bmatrix} m_2 - m_1 \\ \dots \\ m_1 - m_2 \end{bmatrix}, \quad \text{for } (X, \eta) \in W''_{(1)}.$$

We note that $\xi_{||}(X, \eta) = 0$, if $m_1 = m_2$.

Recapitulating the considerations, we now have that the system \mathcal{A} decomposes into two subsystems according to $p_Y(S_A) = W'_{(1)} \cup W''_{(1)}$, and it is not a regular system because $\mathcal{M}_A = p_Y(S_A)$ is the proper subset of W_A .

Let us now analyse the behavior of these trajectories of \mathcal{A} , which start in $W''_{(1)}$ and tend towards $W'_{(1)}$. We consider the following two situations (it has been assumed that $m_2 > m_1$).

Let $Y^0 = (X^0, \eta^0)$ be a point in $W''_{(1)}$ such that $x_1^0 > 0$ and $\eta_1^0 < 0$. Then there must be $\eta_2^0 > 0$, and from Eq. (4.24) one has $\xi_1(Y) < 0$ and $\xi_2(Y) < 0$. Thus both $\eta_1(t)$ and $\eta_2(t)$ decrease along the trajectory of \mathcal{A} starting at Y^0 . The trajectory reaches $W'_{(1)}$ in a finite time where the value of η_1 reduces immediately to zero, and η_2 becomes constant, with the value equal to a .

Also, for Y^0 in $W''_{(1)}$ such that $x_1^0 < 0$ and $\eta_1^0 > 0$, both $\eta_1(t)$ and $\eta_2(t)$ increase in time, and the corresponding trajectory of A reaches $W'_{(1)}$ in a finite time.

In both situations one observes discontinuity in the motion of the system. The particles starting in $W''_{(1)}$ and reaching $W'_{(1)}$ behave as if they had been thrown in the direction of the transporting band moving with a constant speed along the x_2 -axis in R^2_A . \square

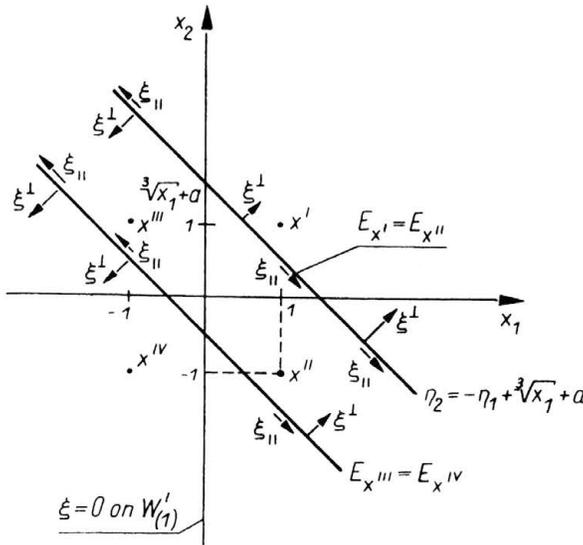


FIG. 4. Geometric structure of the space S_A of the system A — the vector field generated by the system A on its dynamic space $\mathcal{M}_A = W_{(1)}$. It has been assumed that $m_2 > m_1$ and $a = 0.5$.

5. Conclusions

Regularity of nonlinear, nonholonomic mechanical systems was discussed, where the mechanical system A was called a regular one if its dynamic space \mathcal{M}_A (the space of motion of the system) was equal to the entire state space (the extended configuration space) of the system, and there was a well-defined dynamic equation describing the motion of A on \mathcal{M}_A .

This regularity definition applies fully to general dynamic systems, e.g., electrical systems or control systems. The dynamic system \mathcal{D} is said to be a regular one if the dynamic space \mathcal{M} of the system equals the entire (extended) configuration space of the system, and the system has a well-defined dynamic equation on \mathcal{M} .

Regularity has been found to be an important property of a mechanical system because excluding systems which exhibit some kind of discontinuous modes, the well-posed mechanical systems are regular in the sense considered in the paper. The conditions ensuring regularity of a mechanical system have been formulated in Sect. 4.

Nonregular systems have also been discussed. In Theorem 3 (Sect. 3 of the paper) a general procedure for finding the space of motion and the dynamic equation of the

system has been proposed. The procedure has been illustrated in the Example, in Sect. 4, where the nonregular mechanical system is being analysed.

And it was surprising for the author to find that it is not an easy task to find the example of a nonregular mechanical system, such that no discontinuous changes in the x variable, describing the position of the system, are observed. In this context the situation among the electrical networks is quite different. One can find very simple constructions of nonregular electrical networks [7, 8]. Such a construction is the electrical network built of two capacitors connected in the circuit.

Several further generalizations concerning nonregular systems have been included in the Appendix.

The d'Alembert principle was the basis for the considerations. The concept of the d'Alembert space has been introduced and it has been found to be very useful when the problem of finding the dynamic space (the space of motion) and the dynamic equation of a mechanical system with both holonomic and nonholonomic constraints is considered.

Appendix

We will state several facts which are the generalizations of the results in Sect. 3 concerning the spaces of motion and the dynamic equations of nonregular lumped mechanical systems. The concept of (a locally vector-continuous) invariant submanifold is of basic importance in the considerations.

DEFINITION A.1. For fixed $Z^0 = (Y^0, \zeta^0, \xi^0) \in TW_A$, \bar{l}_{z^0} denotes a continuous vector line in TW_A passing through the point Z^0 . This means that:

- j. if $(\zeta^0, \xi^0) = 0$, then $\bar{l}_{z^0} = (Y^0, 0, 0)$, and
- jj. if $(\zeta^0, \xi^0) \neq 0$, then \bar{l}_{z^0} is a one-dimensional connected submanifold in TW_A such that: $Z^0 \in \bar{l}_{z^0}$, $l_{Y^0} \stackrel{\Delta}{=} p_Y(\bar{l}_{z^0})$ is a one-dimensional differentiable submanifold of W_A , l_{z^0} is the graph of a continuous function $l_{Y^0} \ni Y \rightarrow (\zeta, \xi) (Y) \in T_Y l_{Y^0}$, and $(\zeta, \xi) (Y) \neq 0$ for all $Y \in l_{Y^0}$.

(Thus \bar{l}_{z^0} is the vector line of the first kind j — the equilibrium point $(Y^0, 0, 0)$, or it is the vector line of the second kind jj.). \square

In the considerations we need the following property of a subset of the tangent bundle of a differentiable manifold.

DEFINITION A.2. We say that a subset B of TW_A is locally vector-continuous if for each point Z^0 in B there is a continuous vector line in TW_A (the Definition A.1), which is contained in B and passes through the point Z^0 . \square

The projection $p_Y(B)$ of the subset $B \subseteq TW_A$ in Definition A.2 need not have the structure of a differentiable submanifold of W_A .

THEOREM A.1. Let the subset B of the tangent bundle TW_A of the manifold W_A have the following structure: for each point $Y^0 \in p_Y(B)$ there is a differentiable submanifold S of

W_A contained in $p_Y(B)$, such that $Y^0 \in S$ and the set

$$M_S \stackrel{\Delta}{=} B \cap T_S W_A,$$

where $T_S W_A \stackrel{\Delta}{=} \bigcup_{Y \in S} T_Y W_A$, is a C^0 -vector subbundle over S [see [3]] of the tangent bundle TW_A , or M_S is a continuous vector field on S .

Then the subset B is locally vector-continuous.

Proof. Let $Z^0 \in B$, and let S be a differentiable submanifold of W_A contained in $p_Y(B)$ such that $p_Y(Z^0) \in S$ and the set $M_S = B \cap T_S W_A$ has the structure assumed in the text of the theorem. Then $Z^0 \in M_S$ and there is a continuous vector line in M_S passing through the point Z^0 . The conditions which ensure that B is locally vector-continuous are then fulfilled. \square

DEFINITION A.3. The differentiable submanifold H of the extended configuration space W_A of a lumped mechanical system Λ is an invariant submanifold for the system Λ if

$$(A.1) \quad p_Y(S_A \cap TH) = H.$$

The submanifold $H \subseteq W_A$ is the maximal invariant submanifold for Λ if:

1. H is an invariant submanifold for Λ , and
2. H is not a proper subset of the submanifold of W_A , which is invariant for Λ . \square

We have the following theorem:

THEOREM A.2. If the dynamic space \mathcal{M}_A of a lumped mechanical system Λ is the differentiable submanifold of the extended configuration space W_A of Λ , then \mathcal{M}_A is an invariant submanifold for Λ .

Proof. It suffices to prove that for each $Y^0 \in \mathcal{M}_A$, $S_A \cap T_{Y^0} \mathcal{M}_A \neq \phi$.

Let $Y(\cdot)$ be a solution of the system Λ such that $Y(0) = Y^0$. Then $\tilde{D}Y(0) \in S_A \cap T_{Y^0} W_A$, and noting that $\text{Im}(Y(\cdot)) \subseteq \mathcal{M}_A$, $\tilde{D}Y(0) \in S_A \cap T_{Y^0} \mathcal{M}_A$.

Thus $S_A \cap T_{Y^0} \mathcal{M}_A \neq \phi$. \square

DEFINITION A.4. We say that an invariant submanifold H for the system Λ is locally vector-continuous if the subset $S_A \cap TH$ is locally vector-continuous. \square

THEOREM A.3. Let us assume for a lumped mechanical system Λ that every invariant submanifold for Λ is locally vector-continuous.

Then $H \subseteq \mathcal{M}_A$ for every invariant submanifold H for Λ .

Proof. Let H be an invariant submanifold for Λ and let Y^0 be any point in H . It suffices to show that there is a solution of the system Λ passing through the point Y^0 .

From the condition (A.1) defining the invariance of H we have that $S_A \cap T_{Y^0} H \neq \phi$. Let (ζ^0, ξ^0) be a vector in $T_{Y^0} H$ such that $(Y^0, \zeta^0, \xi^0) \in S_A \cap T_{Y^0} H$. By the local vector-continuity of H there is a continuous vector line \bar{l}_0 in $S_A \cap TH$ passing through the point (Y^0, ζ^0, ξ^0) . For the differential equation $dY/dt = f(Y)$, defined on the differentiable submanifold $l_0 = p_Y(\bar{l}_0)$ in H and corresponding the vector line \bar{l}_0 , there is a uniquely defined solution $\bar{Y}(\cdot)$ satisfying the initial condition $\bar{Y}(0) = Y^0$, and $\bar{Y}(\cdot)$ is a part of some

solution of the system $\Lambda(\bar{Y}(\cdot))$ satisfies the conditions specified in Definition 3, except the maximality condition on $\text{Dom}(\bar{Y}(\cdot))$. Hence $Y^0 \in \mathcal{M}_\Lambda$. \square

THEOREM A.4. *Let us assume for the system Λ that every invariant submanifold for Λ is locally vector-continuous.*

Then the dynamic space \mathcal{M}_Λ of Λ is the set union of all invariant submanifolds of W_Λ for the system Λ .

PROOF. By Theorem A.3, the set union of all invariant submanifolds for the system Λ is the subset of the dynamic space of Λ .

Let Y^0 be a point in \mathcal{M}_Λ . There exists a solution $Y(\cdot)$ of the system Λ such that $Y(t_0) = Y^0$, for some $t_0 \in \text{Dom}(Y(\cdot))$, and such that $\text{Im}(Y(\cdot)|_{\Delta t})$ is a differentiable submanifold of W_Λ for some open interval Δt in R_t containing t_0 . Since $\text{Im}(Y(\cdot)|_{\Delta t})$ is an invariant submanifold for the system Λ , we have proved that each point $Y^0 \in \mathcal{M}_\Lambda$ belongs to some invariant submanifold for Λ . This completes the proof. \square

THEOREM A.5. *Let us assume for the lumped mechanical system Λ , that every invariant submanifold for the system Λ is locally vector-continuous, and let the dynamic space \mathcal{M}_Λ of Λ be a differentiable submanifold of W_Λ .*

Then the dynamic space \mathcal{M}_Λ of Λ is the (unique) maximal invariant submanifold for the system Λ .

Additionally, if the system Λ generates a vector field $f_\Lambda(\cdot)$ on its dynamic space \mathcal{M}_Λ , then

$$\tilde{f}_\Lambda = S_\Lambda \cap T\mathcal{M}_\Lambda.$$

PROOF. From Theorem A.2 it follows that the submanifold \mathcal{M}_Λ is an invariant submanifold for the system Λ . Noting then that the assumptions of the theorem assure that every invariant submanifold for the system Λ is a subset of \mathcal{M}_Λ , we obtain the first part of the thesis.

We shall prove the second part of the thesis.

Fix a point $Z^0 \in S_\Lambda \cap T\mathcal{M}_\Lambda$. By the invariance of the submanifold \mathcal{M}_Λ and the assumed local vector-continuity of every invariant submanifold of W_Λ for the system Λ , there is a continuous vector line \bar{l}_0 contained in $S_\Lambda \cap T\mathcal{M}_\Lambda$ and passing through the point Z^0 .

For the differential equation $dY/dt = f(Y)$, defined on the differentiable submanifold $l_0 = p_Y(\bar{l}_0)$ in $p_Y(S_\Lambda \cap T\mathcal{M}_\Lambda)$ and the corresponding vector line \bar{l}_0 , we have a uniquely defined solution $\bar{Y}(\cdot)$ satisfying the initial condition $\bar{Y}(0) = Y^0$, $Y^0 = p_Y(Z^0)$. $\bar{Y}(\cdot)$ is a part of some solution of the system Λ . And next, from the assumption that the system generates a vector field on its dynamic space \mathcal{M}_Λ (the vector field $f_\Lambda(\cdot)$), we obtain that there is a continuous vector line \bar{l} in \tilde{f}_Λ , such that $\bar{l} = \text{Im}(\bar{D}\bar{Y}(\cdot))$. Thus $Z^0 = \bar{D}\bar{Y}(0) \in \tilde{f}_\Lambda$ and hence $S_\Lambda \cap T\mathcal{M}_\Lambda \subseteq \tilde{f}_\Lambda$. On the other hand, from Definition 5 of the vector field generated by a lumped mechanical system Λ we have that $\tilde{f}_\Lambda \subseteq S_\Lambda \cap T\mathcal{M}_\Lambda$. Finally then, $\tilde{f}_\Lambda = S_\Lambda \cap T\mathcal{M}_\Lambda$. \square

In the following Theorem A.6, being the extension of Theorem 3 of Sect. 3 of the paper, a general multi-step reduction procedure is proposed, which enables to exclude these points in W_Λ , which are not in the space of motion of the system Λ . Let us note that Theorem 3 concerns only the case when $\mathcal{M}_\Lambda = p_Y(S_\Lambda)$ and $\tilde{f}_\Lambda = S_\Lambda$.

We consider a lumped mechanical system $\Lambda = (\mathcal{P}_\Lambda, m_\Lambda, W_\Lambda, F_\Lambda(\cdot))$. Set $\tilde{W}_{(0)} = W_\Lambda$ and $\kappa_{(0)}^A = S_\Lambda$.

THEOREM A.6. *Assume that the following holds for the system Λ .*

1. $\tilde{W}_{(1)} \stackrel{\Delta}{=} p_Y(\kappa_{(0)}^A)$ is the differentiable submanifold of W_Λ .

$$\text{Set } \kappa_{(1)}^A = S_\Lambda \cap T\tilde{W}_{(1)}.$$

2. $\tilde{W}_{(2)} \stackrel{\Delta}{=} p_Y(\kappa_{(1)}^A)$ is the differentiable submanifold of W_Λ .

$$\text{Set } \kappa_{(2)}^A = S_\Lambda \cap T\tilde{W}_{(2)}.$$

3. For each $k \geq 3$, $\tilde{W}_{(k)} \stackrel{\Delta}{=} p_Y(\kappa_{(k-1)}^A)$ is the differentiable submanifold of W_Λ .

$$\text{Set } \kappa_{(k)}^A = S_\Lambda \cap T\tilde{W}_{(k)}.$$

Then, if for some k $\kappa_{(k)}^A$ is the C° -vector field on the submanifold $\tilde{W}_{(k)}$ and if the differential equation $dY/dt = \bar{f}_{(k)}(Y)$ corresponding to the vector field $\kappa_{(k)}^A$ on $\tilde{W}_{(k)}$ has unique solutions, then $\tilde{W}_{(k)}$ is the dynamic space of the system Λ and

$$\tilde{f}_\Lambda := \bigcup_{Y \in \tilde{W}_{(k)}} (Y, \bar{f}_{(k)}(Y)) = \kappa_{(k)}^A$$

is the vector field generated by the system Λ on its dynamic space $\mathcal{M}_\Lambda = \tilde{W}_{(k)}$.

P r o o f. Note that $\tilde{W}_{(k)} \subseteq W_\Lambda$. Since each solution $Y(\cdot) = Y(Y^0, \cdot)$ of the differential equation $dY/dt = \bar{f}_{(k)}(Y)$ (defined on $\tilde{W}_{(k)}$) satisfies the relation (2.6) for each $t \in \text{Dom}(Y(\cdot))$, then each of them is a part of some solution of the system Λ . It follows then that $\tilde{W}_{(k)} \subseteq \mathcal{M}_\Lambda$.

We shall now prove that $\mathcal{M}_\Lambda \subseteq \tilde{W}_{(k)}$.

Fix a point $Y^0 \in \mathcal{M}_\Lambda \subseteq W_\Lambda$. It follows from the definition of the dynamic space of the mechanical system Λ that there exists a solution $Y(\cdot)$ of Λ passing through Y^0 at $t = 0$ and, by assumption, the function $Y(\cdot)$ is differentiable. Thus we have a well-defined vector $\bar{f}^0 = \left(\frac{d}{dt} Y \right) (0)$ which is tangent to W_Λ at the point Y^0 and by the definition of the solution of the mechanical system, $(Y^0, \bar{f}^0) \in S_\Lambda$.

It follows then that $Y^0 \in \tilde{W}_{(1)}$ and hence, $\mathcal{M}_\Lambda \subseteq \tilde{W}_{(1)}$.

Using the same arguments, we successively obtain that $\mathcal{M}_\Lambda \subseteq \tilde{W}_{(2)}$, $\mathcal{M}_\Lambda \subseteq \tilde{W}_{(3)}$, ..., and $\mathcal{M}_\Lambda \subseteq \tilde{W}_{(k)}$.

Finally we then have that $\mathcal{M}_\Lambda \subseteq \tilde{W}_{(k)}$ and hence $\mathcal{M}_\Lambda = \tilde{W}_{(k)}$. (Observe also that the spaces $\tilde{W}_{(k+r)}$ and $\kappa_{(k+r)}^A$, $r = 1, 2, \dots$, are identical with the spaces $\tilde{W}_{(k)}$ and $\kappa_{(k)}^A$, respectively).

Noting that: the solutions of $dY/dt = \bar{f}_{(k)}(Y)$ are defined uniquely, each solution of $dY/dt = \bar{f}_{(k)}(Y)$ is the solution of the system Λ and each solution of Λ satisfies the differential equation $dY/dt = \bar{f}_{(k)}(Y)$, we obtain that: the solutions of $dY/dt = \bar{f}_{(k)}(Y)$ define a flow on $\tilde{W}_{(k)}$ and $\tilde{f}_\Lambda := \kappa_{(k)}^A$ is the vector field generated by the mechanical system Λ on its dynamic space $\mathcal{M}_\Lambda = \tilde{W}_{(k)}$. \square

Observe that $k = \text{Dim } W_A + 1$ is the maximal number of steps in the procedure described in Theorem A.6 in the case when it is effective.

Let us also observe that for a lumped mechanical system $A = (\mathcal{P}_A, m_A, W_A, F_A(\cdot))$, the multi-step procedure described in the text of Theorem A.6 always ends at the first step, if $S_A \cong T(p_Y(S_A))$, in every case when it is effective.

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INSTITUTE OF ELECTRONIC TECHNOLOGY
TECHNOLOGICAL UNIVERSITY OF GDAŃSK.

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