

Fracture effects in the propagation of a shock wave through a bulk solid

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THE APPROACH of nonlinear continuum mechanics is used in formulating three-dimensional stress-strain and internal state variable constitutive equations for a class of bulk solids. The internal state variable influences the generalized elastic modulus and describes the sensitivity of the rheological material to changes of the specific volume. A plane shock wave propagating through the material medium is analysed. An effect of fracture characterized by the blow-up of the amplitude of the wave in finite time, is discussed. The conditions for such a behaviour of the wave are formulated.

Wykorzystano koncepcję nieliniowej mechaniki kontinuum przy formułowaniu trójwymiarowych równań konstytutywnych: naprężenie-odkształcenie i wewnętrzna zmienna stanu, dla klasy ciał sypkich i skałopodobnych. Zmienna wewnętrzna (parametr wewnętrzny), wpływając na uogólniony moduł sprężystości, opisuje czułość materiału na zmianę objętości właściwej. Analizuje się płaską falę uderzeniową rozprzestrzeniającą się w takim ośrodku materialnym. Rozważa się efekt zniszczenia scharakteryzowany gwałtownym (nieskończonym) wzrostem w skończonym czasie amplitudy fali. Sformułowano warunki takiego zachowania się fali.

Использован подход нелинейной механики континуум при формулировке трехмерных определяющих уравнений: напряжение — деформация и внутренняя переменная состояния для класса сыпучих и скалоподобных тел. Внутренняя переменная (внутренний параметр), влияя на обобщенный модуль упругости, описывает чувствительность материала на изменение удельного объема. Анализируется плоская ударная волна распространяющаяся в такой материальной среде. Рассматривается эффект разрушения охарактеризованный внезапным (бесконечным) ростом в конечном отрезке времени амплитуды волны. Сформулированы условия такого поведения волны.

1. Introduction

DURING the past few years there have been considerable theoretical studies on constitutive equations for non-metallic solids and on the propagation of waves in viscoelastic solids. However, this research has not really taken into account the fact that many non-metallic solids present in nature exhibit a moderate change of values of elasticity moduli during non-isochoric deformations. These properties are known to have a significant influence on the flow kinetics of granular media and it has been found that in most of the bulk solids the slope and curvature of unloading curves vary depending on the history of previous deformation [1-4].

In most of the theoretical studies granular and rockline materials are described under the assumptions of the infinitesimal theory of elastic-plastic solids. The flow rules used there appear to be a kind of non-associated elastic-plastic laws with variable elasticity represented by a family of matrices parametrized with respect to plastic density change (cf. [5]) or some plastic parameter [6].

In these theoretical studies the framework of nonlinear continuum mechanics has not often been used. Moreover the constitutive equations proposed for soils, rocks and other granular media have rather been written in stress-free configurations. But it seems to be more adequate to formulate the equations in prestressed states, i.e. with some non-vanishing and nonhomogeneous (in general) hydrostatic pressures. Those states are present in nature, whereas the stress-free configurations are the experimenter's idealisations.

In this paper the nonlinear continuum mechanics approach is used in formulating three-dimensional stress-strain and internal state variable constitutive equations. The equations derived take into account the existence of the hydrostatic (nonhomogeneous) pressure in the reference configuration. A simple model is considered, in which one scalar internal state variable depends on the history of the specific volume changes. For those relations a plane shock wave is analysed. The amplitude of the wave satisfies a Bernoulli equation. Its general solution is discussed. An effect of fracture characterized by the blow-up of the amplitude in the finite time is discussed. The conditions for such behaviour of the wave are given in terms of the initial amplitude and the material functions.

2. Bulk medium as a rheological continuum sensitive to specific volume changes

In such solids like rocks, soils and ceramic grains the history of deformation strongly affects the current elastic moduli. Moreover, these media demonstrate large changes in the specific volume due to loadings. One can also observe different time phenomena like aging, relaxation and creep. These effects arise in dynamics as well as in quasi-statics. Consequently, our attention will be directed to the constitutive description of those solids in the framework of finite deformations used as a suitable rheological model.

We think that *the internal state variable approach* is more convenient in this respect. Taking a general isotropic *constitutive equation* for the Cauchy stress $\tilde{\mathbf{T}}$ in the theory with internal (scalar) state variables $\alpha_1, \dots, \alpha_r$, we can write under isothermal conditions

$$(2.1) \quad \tilde{\mathbf{T}} = \psi_0 \mathbf{1} + \psi_1 \mathbf{B} + \psi_2 \mathbf{B}^2,$$

where $\mathbf{B} = \mathbf{F} \mathbf{F}^T$ is the left Cauchy-Green strain tensor and \mathbf{F} is the deformation gradient. The quantities ψ_0, ψ_1, ψ_2 are assumed to be functions of $I_{\mathbf{B}}, II_{\mathbf{B}}, III_{\mathbf{B}}$ invariants of \mathbf{B} and internal state variables.

According to our viewpoint in the reference configuration, i.e. $\mathbf{B} = \mathbf{1}$, the medium is in a state of hydrostatic pressure, i.e. $\tilde{\mathbf{T}} = -p \mathbf{1}$. In rocklike media the pressure p depends in general on the depth, i.e. $p = p(\mathbf{X})$. This effect can be described by the nonhomogeneous distribution of the internal state variables $\alpha_{(0)m}, m = 1, 2, \dots, r$, in the reference configuration. This means that if the pair $(\mathbf{B}, \alpha_m) = (\mathbf{1}, \alpha_{(0)m})$ determines a state of the medium in the reference configuration, then the $\alpha_{(0)m}$ are functions of \mathbf{X} , in general.

Assume for simplicity that

$$(2.2) \quad \begin{aligned} \psi_0 &= \psi_2 \equiv 0, \\ \psi_1 &= \psi_1(III_{\mathbf{B}}, \alpha) = (III_{\mathbf{B}})^{-\frac{1}{2}} \psi(\alpha). \end{aligned}$$

This means that we confine the number of variables to one as well as the number of the material functions. Since $\text{III}_B = \det \mathbf{B} = \det \mathbf{F} \det \mathbf{F}^T = (\det \mathbf{F})^2$ and

$$(2.3) \quad \rho_0/\rho = \det \mathbf{F},$$

the assumption (2.2) leads to the following stress constitutive equation

$$(2.4) \quad \tilde{\mathbf{T}} = \frac{\rho}{\rho_0} \psi(\alpha) \mathbf{B}.$$

For this form of equation the pressure in the reference configuration will be given by

$$(2.5) \quad p = -\psi(\alpha).$$

Note that in Eq. (2.4) the function $\psi(\alpha)$ plays the role of the elastic modulus depending on time and the history of deformation. In fact, a general *evolution equation* for the internal state variable α can be written in the form

$$\dot{\alpha} = \hat{a}(\alpha, \text{I}_B, \text{II}_B, \text{III}_B),$$

where $\dot{\alpha}$ denotes the material time derivative of α . The variable α has to describe the sensitivity of the material to specific volume changes, consequently it seems natural to assume that \hat{a} depends only on $(\text{III}_B)^{1/2}$ and α . Taking the linear form of this dependency

$$(2.6) \quad \hat{a}(\alpha, (\text{III}_B)^{1/2}) = -h_0 \alpha + ((\text{III}_B)^{1/2} - 1) h_1,$$

where $h_0 > 0$ and h_1 are constants, we may write the final form of the evolution equation as follows

$$(2.7) \quad \text{or} \quad \begin{aligned} \dot{\alpha} &= -h_0 \alpha + (\det \mathbf{F} - 1) h_1, \\ \dot{\alpha} &= -h_0 \alpha + (\rho_0/\rho - 1) h_1. \end{aligned}$$

For further analysis it is more convenient to use the first Piola-Kirchhoff stress tensor \mathbf{T} given by

$$(2.8) \quad \mathbf{T} = \det(\mathbf{F}) \tilde{\mathbf{T}} (\mathbf{F}^T)^{-1}.$$

With Eq. (2.8) the constitutive equation (2.4) takes the form

$$(2.9) \quad \mathbf{T} = \psi(\alpha) \mathbf{F}.$$

The dependence of the modulus ψ on α should be determined experimentally as well as the sign of the constant h_1 . The assumed positive sign of h_0 in Eq. (2.7) is motivated by the stability requirement of the solution of Eq. (2.7).

Let us notice that during an isochoric deformation (i.e. with $\det \mathbf{F} = 1$) Eqs. (2.7) and (2.9) describe a medium with variable elastic modulus in time.

3. Compatibility conditions and the speed of propagation of a plane shock wave

Consider a plane shock wave propagating through the material medium in the X_1 -direction (the Cartesian coordinate system is assumed). This means that a moving singular surface $\{\Sigma(t)\}_{t \in I}$ (where I is a time interval $[0, t_f]$), across which the first derivatives of

the function of motion χ suffer jump discontinuities, forms a family of parallel planes with the unit normal

$$(3.1) \quad \mathbf{N} = [1, 0, 0].$$

According to this assumption all functions taking place in the description depend on X_1 and t , only. The general *kinematical, geometrical and dynamical compatibility conditions* (cf. [7, 8]) at the shock wave propagating with the positive speed U_N are

$$(3.2) \quad \begin{aligned} U_N[\mathbf{F}] &= -[\mathbf{v}] \otimes \mathbf{N}, \\ 2\sqrt{U_N} \frac{\delta}{\delta t} (\sqrt{U_N} [\mathbf{F}]\mathbf{N}) &= U_N^2 [\text{Grad} \mathbf{F}] \cdot (\mathbf{N} \otimes \mathbf{N}) - [\dot{\mathbf{v}}], \\ [\mathbf{T}]\mathbf{N} &= -\rho_0 U_N [\mathbf{v}], \end{aligned}$$

where the bracket $[\]$ denotes the jump, i.e. for any $f(\mathbf{X}, t)$ the formula

$$(3.3) \quad [f](\mathbf{Y}, t) = \lim_{\substack{k \rightarrow 0 \\ k > 0}} f(\mathbf{Y} - k\mathbf{N}, t) - \lim_{\substack{k \rightarrow 0 \\ k > 0}} f(\mathbf{Y} + k\mathbf{N}, t), \quad \mathbf{Y} \in \Sigma(t),$$

gives the jump at $\Sigma(t)$, $t \in I$. Note that

$$f^-(\mathbf{Y}, t) = \lim_{\substack{k \rightarrow 0 \\ k > 0}} f(\mathbf{Y} - k\mathbf{N}, t)$$

is the limiting value of f taken from the region immediately behind the wave. The second term on the right side of Eq. (3.3) is the limiting value f^+ of f immediately ahead of the wave. The so-called displacement (or THOMAS' [9]) derivative

$$(3.4) \quad \frac{\delta(\cdot)}{\delta t} = \frac{\partial(\cdot)}{\partial t} + U_N \text{Grad}(\cdot) \cdot \mathbf{N}$$

measures the rate of change of any quantity along the normal trajectories of the wave.

Here $\mathbf{v} = \partial\chi/\partial t$ is the particle velocity vector and $\dot{\mathbf{v}}$ is the acceleration vector.

At the plane wave, these conditions lead to

$$(3.5) \quad \begin{aligned} U_N[F_{i1}] &= -[v_i], \quad [F_{kj}] = 0, \quad i, k = 1, 2, 3, \quad j = 2, 3, \\ [T_{i1}] &= -\rho_0 U_N [v_i], \end{aligned}$$

$$2\sqrt{U_N} \frac{\delta}{\delta t} (\sqrt{U_N} [F_{i1}]) = U_N^2 [F_{i1,1}] - [\dot{v}_i].$$

Away from $\{\Sigma(t)\}$ the first equation of motion in the reference configuration

$$(3.6) \quad \text{Div} \mathbf{T} + \rho_0 \mathbf{b} = \rho_0 \mathbf{v}$$

holds, while across $\{\Sigma(t)\}$, with the continuous body force \mathbf{b} ,

$$(3.7) \quad [\text{Div} \mathbf{T}] = \rho_0 [\dot{\mathbf{v}}].$$

Putting Eq. (3.7) into Eq. (3.2)₂ the following general equation can be obtained:

$$(3.8) \quad 2\sqrt{U_N} \frac{\delta}{\delta t} (\sqrt{U_N} [\mathbf{F}]\mathbf{N}) = U_N^2 [\text{Grad} \mathbf{F}] \cdot (\mathbf{N} \otimes \mathbf{N}) - \frac{1}{\rho_0} [\text{Div} \mathbf{T}].$$

Because of Eq. (3.1) Eq. (3.8) (compare Eq. (3.5)₃) reduces to

$$(3.9) \quad 2\sqrt{U_N} \frac{\delta}{\delta t} (\sqrt{U_N} [F_{i1}]) = U_N^2 [F_{i1,1}] - \frac{1}{\rho_0} [T_{i1,1}].$$

The speed of propagation U_N can be derived from Eq. (3.5)_{1,2} as follows:

$$(3.10) \quad \rho_0 U_N^2 = \frac{[T_{i1}]}{[F_{i1}]}.$$

The following relations $[T_{11}]/[F_{11}] = [T_{21}]/[F_{21}] = [T_{31}]/[F_{31}]$ are the simple consequences of Eq. (3.10); they give the restrictions on the jump of the deformation gradient, indicating that it cannot be prescribed arbitrarily.

The constitutive equation (2.9) implies the following condition:

$$(3.11) \quad [T_{i1}] = [\psi(\alpha) F_{i1}].$$

Now the question arises whether the equality $[\psi(\alpha) F_{i1}] = \psi(\alpha) [F_{i1}]$ is true. In fact, it is. It can be shown, using the generalized Rankine-Hugoniot condition (cf. [10]), that the internal variable α satisfying Eq. (2.7) in the points of differentiability on the wave fulfils the relation⁽¹⁾

$$(3.12) \quad U_N [\alpha] = 0.$$

This means that α is continuous across the wave. Hence continuity of the function ψ implies

$$(3.13) \quad [\psi(\alpha)] = 0.$$

The relations (3.10)–(3.13) lead to the following expression for the speed of propagation

$$(3.14) \quad \rho_0 U_N^2 = \psi(\alpha).$$

This expression reminds the speed equation for elastic waves. The main difference lies in the dependence of the generalized elastic modulus ψ on the variables α . This will affect the growth of the wave amplitude.

4. The amplitude equation

In this section the necessary calculation are performed in order to find an equation which governs the growth and decay of the jump in \mathbf{F} on the wave.

The jump in the derivative $[T_{i1,1}]$ appearing in Eq. (3.9) can be calculated using the chain rule property

$$[T_{i1,1}] = \left\| \frac{\partial \mathcal{F}_{i1}}{\partial F_{kL}} F_{kL,1} \right\| + \left\| \frac{\partial \mathcal{F}_{i1}}{\partial \alpha} \alpha_{,1} \right\|,$$

where $\mathcal{F}_{i1}(F_{kL}, \alpha) = \psi(\alpha) F_{i1}$. Hence $\frac{\partial \mathcal{F}_{i1}}{\partial F_{kL}} = \delta_{ik} \delta_{1L} \psi(\alpha)$

and

$$(4.1) \quad [T_{i1,1}] = \psi(\alpha) [F_{i1,1}] + \psi'(\alpha) [\alpha_{,1} F_{i1}].$$

⁽¹⁾ The proof of this result is similar to the one-dimensional case considered in [11].

Putting Eq. (4.1) into Eq. (3.9), after reduction, and with the help of Eq. (3.14), we have

$$(4.2) \quad 2\sqrt{U_N} \frac{\delta}{\delta t} (\sqrt{U_N} [F_{i1}]) = -\frac{\psi'(\alpha)}{\varrho_0} [\alpha_{,1} F_{i1}].$$

The general kinematical compatibility condition (cf. [7-9])

$$[\dot{f}] = -U_N [\text{Grad} f] \mathbf{N}$$

for any continuous function f on the wave gives for the derivatives of the function α the relation

$$(4.3) \quad [\alpha_{,1}] = -\frac{1}{U_N} [\dot{\alpha}]$$

which can be used in the further manipulations of Eq. (4.2).

From the definition of the jump discontinuity, the product of two quantities f and g has the following jump discontinuities:

$$(4.4) \quad [fg] = [f][g] + f^+[g] + [f]g^+.$$

Applying Eqs. (4.3) and (4.4) to the last term in Eq. (4.2) we may write

$$(4.5) \quad [\alpha_{,1} F_{i1}] = -\frac{[\dot{\alpha}]}{U_N} [F_{i1}] + (\alpha_{,1})^+ [F_{i1}] - \frac{[\dot{\alpha}]}{U_N} F_{i1}^+.$$

From the evolution equation (2.7) the jump in $\dot{\alpha}$ is

$$(4.6) \quad [\dot{\alpha}] = h_1 [\det \mathbf{F}].$$

Hence the full equation for the amplitude $A_i := [F_{i1}]$ of the wave will be

$$(4.7) \quad 2\sqrt{U_N} \frac{\delta}{\delta t} (\sqrt{U_N} A_i) = \frac{\psi'(\alpha)}{\varrho_0} \left\{ \frac{h_1}{U_N} [\det \mathbf{F}] A_i - (\alpha_{,1})^+ A_i + \frac{h_1}{U_N} [\det \mathbf{F}] F_{i1}^+ \right\}.$$

We now make the type of deformation ahead of the wave precise by the following assumption:

The medium ahead of the wave (i.e. in + region) is in an equilibrium (general nonhomogeneous) state with the Cartesian axes X_1, X_2, X_3 being the principal axes.

The above assumption corresponds to that adopted by BLAND [12]. From the latter the following relations hold:

$$(4.8) \quad \begin{aligned} \dot{\alpha}^+ &= -h_0 \alpha + ((\det \mathbf{F})^+ - 1) h_1 = 0, \\ [F_{ik}^+] &= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \end{aligned}$$

$$\psi'(\alpha)(\alpha_{,1})^+ \lambda_1 + \psi(\alpha) \lambda_{1,1} + \varrho_0 b_1 = 0,$$

where $(\det \mathbf{F})^+ = \lambda_1 \lambda_2 \lambda_3$. The first relation in Eq. (4.8) is the evolution equation for the variable α in the "+" region. The third is the equilibrium equation which can be used in the determination of $(\alpha_{,1})^+$ in terms of $\lambda_1, \lambda_{1,1}$ and the body force component b_1 .

In view of Eq. (4.8)₁ the jump in $\det \mathbf{F}$ is given by

$$(4.9) \quad [\det \mathbf{F}] = A_1 \lambda_2 \lambda_3.$$

Let us calculate $(\delta/\delta t)U_N$

$$(4.10) \quad \frac{\delta U_N}{\delta t} = \frac{\delta}{\delta t} (\varrho_0^{-1} \psi(\alpha))^{1/2} = \frac{1}{2} (\varrho_0^{-1} \psi'(\alpha)(\alpha_{,1})^+ - \varrho_0^{-2} \psi(\alpha) \varrho_{0,1}),$$

where we have used the definition of $\delta/\delta t$ together with Eqs. (3.1) and (4.8)₁. From Eq. (4.10) we can see that the nonhomogeneous distribution of α in the medium ahead of the wave implies the non-constant speed of propagation U_N . In a linear elastic solid the vanishing of $\varrho_{0,1}$ (i.e. the homogeneous mass distribution) has led to constant U_N .

Applying Eqs. (4.9) and (4.10) to Eq. (4.8) we have, after some lengthy manipulations,

$$(4.11) \quad \frac{\delta A_i}{\delta t} = \frac{\psi'(\alpha) h_1 \lambda_2 \lambda_3}{2\psi(\alpha)} (A_1 A_i + A_1 \lambda_1 \delta_{i1}) + A_i \left(\frac{\psi(\alpha)}{4\varrho_0^2 U_N} \varrho_{0,1} - \frac{3\psi'(\alpha)}{4\varrho_0 U_N} (\alpha_{,1})^+ \right),$$

where U_N is given by Eq. (3.14). This is an amplitude equation of the shock wave. (For the corresponding equation in the one-dimensional theory with internal state variables see [13]; in materials with memory, plane waves were discussed in [14]).

5. Fracture effect

The obtained amplitude equation (4.11) is a set of three coupled Bernoulli equations

$$(5.1) \quad \begin{aligned} \frac{dA_1}{dt} &= \beta A_1^2 + (\beta \lambda_1 + \gamma) A_1, \\ \frac{dA_k}{dt} &= \beta A_1 A_k + \gamma A_k, \quad k = 2, 3, \end{aligned}$$

where⁽²⁾

$$(5.2) \quad \begin{aligned} \beta &= \frac{\psi'(\alpha) h_1 \lambda_2 \lambda_3}{2\psi(\alpha)}, \quad \gamma = \frac{\psi(\alpha)}{4\varrho_0^2 U_N} \varrho_{0,1} - \frac{3\psi'(\alpha)}{4\varrho_0 U_N} (\alpha_{,1})^+, \\ \frac{d}{dt} &= \frac{\partial}{\partial t} + U_N \frac{\partial}{\partial X_1}. \end{aligned}$$

The symbol d/dt is a derivative with respect to t treated as a parameter of the normal trajectory of the wave. The general solutions of Eq. (4.12) with the initial values $A_1(0) \neq 0$ and $A_k(0)$ are

$$(5.3) \quad \begin{aligned} A_1(t) &= \frac{\exp\left(\int_0^t (\beta(\tau) \lambda_1(\tau) + \gamma(\tau)) d\tau\right)}{\frac{1}{A_1(0)} - \int_0^t \beta(\tau) \exp\left(\int_0^\tau \beta(s) \lambda_1(s) + \gamma(s) ds\right) d\tau}, \\ A_k(t) &= A_k(0) \exp\left(\int_0^t (\beta(\tau) A_1(\tau) + \gamma(\tau)) d\tau\right), \quad k = 2, 3. \end{aligned}$$

⁽²⁾ Note that β and γ have the dimension s^{-1} and λ_1 is dimensionless, like the strain.

It is a known fact that solutions of Bernoulli's equation may have some singularities. Especially one observes a solution which becomes unbounded in finite time.

In studies on growth and decay of acceleration waves in various types of nonlinear material media it is found that the amplitudes of these waves obey the Bernoulli equation. The analysis above shows that the amplitude of the shock wave in a solid with variable elastic properties also satisfies the Bernoulli equation.

In two papers BAILEY and CHEN [15] considered the local and global behaviour of the solutions of the Bernoulli equation. BAILEY and CHEN (see Sect. 4 in [15]) showed, in particular, that when the coefficients β , γ , λ_1 satisfy some integrability and boundedness conditions, and

$$(5.4) \quad A_1(0)\beta(t) > 0, \quad \text{for all } t,$$

and

$$(5.5) \quad \text{if } |A_1(0)| > \omega, \quad \text{then } \lim_{t \rightarrow t_\infty} |A_1(t)| = \infty,$$

where ω is the critical initial amplitude given by

$$(5.6) \quad \omega = \left\{ \int_0^\infty |\beta(t)| \exp\left(\int_0^t (\beta(s)\lambda_1(s) + \gamma(s)) ds\right) dt \right\}^{-1},$$

and t_∞ is a finite time defined by

$$(5.7) \quad \int_0^{t_\infty} \beta(t) \exp\left(\int_0^t (\beta(\tau)\lambda_1(\tau) + \gamma(\tau)) ds\right) dt = \frac{|1}{A_1(0)}.$$

The concept of the critical amplitude was first introduced by COLEMAN and GURTIN [16] for the case of one-dimensional acceleration waves in materials with memory with constant coefficients of the Bernoulli equation. Some extended results of Bailey and Chen can be found in the papers BOWEN and CHEN [17] and KOSIŃSKI [18]. The latter deals with acceleration waves in materials with internal state variables.

The unbounded growth of the amplitudes A_t of the shock wave means that the F_{i1}^- grow without bound, for

$$(5.8) \quad A_t := \llbracket F_{i1} \rrbracket = F_{i1}^- - F_{i1}^+$$

and the F_{i1}^+ are finite while the values ahead of the wave.

This phenomenon can be interpreted as a fracture of the material (at this point and time, i.e. at $Y \in \Sigma(t_\infty)$, (cf. Sect. 2). The quantities F_{i1} are the gradients of the u_i components of the displacement vector \mathbf{u} . The unbounded growth, at the same points, of the gradients can be viewed as a symptom of the local continuity loss of the medium. In terms of the stress components $\llbracket T_{i1} \rrbracket$ (cf. Eq. (3.11)) their unbounded growth in finite time means that some (and any) stress fracture criterion of the material can be reached.

The inequality (5.4)₁ may be satisfied in two cases. The first case is, in view of Eqs. (4.9) and (5.2)₁,

$$(5.9) \quad \frac{\psi'(\alpha)h_1}{2\psi(\alpha)} \llbracket \det \mathbf{F} \rrbracket > 0 \quad \text{and} \quad \llbracket \det \mathbf{F} \rrbracket < 0.$$

This means that $\det \mathbf{F}^- < \det \mathbf{F}^+$ and the wave is compressive relative to the "+" region. On the other hand the universal inequality $0 < \det \mathbf{F}$ applied to Eq. (5.9)₂ implies

$$-\det \mathbf{F}^+ < \det \mathbf{F}^- < \det \mathbf{F}^+.$$

Since $\det \mathbf{F}^+$ is finite, the compressive wave cannot lead to any unbounded growth of A_1 .

The second case is the expansive wave with

$$(5.10) \quad [\det \mathbf{F}] > 0 \quad \text{and} \quad \frac{\psi'(\alpha)h_1}{2\psi(\alpha)} [\det \mathbf{F}] > 0.$$

This implies $\psi'(\alpha)h_1/2\psi(\alpha) > 0$. Hence either

$$(5.11) \quad A_1 < 0 \quad \text{and} \quad \lambda_2 \lambda_3 < 0 \quad \text{if} \quad \beta < 0$$

or

$$(5.12) \quad A_1 > 0 \quad \text{and} \quad \lambda_2 \lambda_3 > 0 \quad \text{if} \quad \beta > 0.$$

If the relations (5.11) are true, then necessarily $\lambda_1 < 0$ and hence $\lambda_1 \beta > 0$. When Eqs. (5.12) are true, then $\lambda_1 > 0$, and $\lambda_1 \beta > 0$, as previously.

Unfortunately, without knowing the state ahead of the wave we cannot, in general, calculate the value of the critical initial amplitude ω . In the case of a homogeneous state $\omega = 0$ (because $\rho_{0,1} = 0$ and $\alpha_{,1} = 0$) and $\beta = \text{const}$, $\beta \lambda_1 = \text{const}$. Then the value of ω is zero and we have without detailed calculation the result:

PROPOSITION 1. *If the equilibrium state ahead of the wave is homogeneous, then any expansive (with respect to this state) wave leads to fracture.*

For a general case, however, we can formulate

PROPOSITION 2. *If the equilibrium state ahead of the wave is nonhomogeneous, then an expansive wave will lead to fracture provided that the initial value of the amplitude A_1 exceeds the critical one, that is,*

$$|A_1(0)| = |F_{11}^-(0) - F_{11}^+(0)| > \omega,$$

where ω is given by Eq. (5.6) while the time t_∞ of fracture is defined by Eq. (5.7).

It should be interesting to analyse particular forms of the function $\psi(\alpha)$ and the constant h_1 , and to estimate the quantity ω in a nonhomogeneous case. The discussion of spherical waves would be particularly significant, especially in rocklike media. But these problems will be treated elsewhere.

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