

Evolution model of the elastic domain by any loading

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IN THE FIELD of elastic-plastic and elastic-viscoplastic theories dealing with finite and small deformations, we offer a hardening model generalizing several current existing models (isotropic, kinematic, ...) and adapted to the modelization of anisotropy which is related to the loading process. The model proposed uses a hardening variable which is an R -valued application (one more variable compared to kinematic and isotropic hardening). A particular aspect of the model has been used in an experiment on an aluminium alloy undergoing complex changes and small plastic strains.

Spośród teorii sprężystoplastyczności i sprężystolepkosprężystości opisujących stany małych i skończonych odkształceń proponuje się w pracy model wzmocnienia stanowiący uogólnienie kilku istniejących modeli (izotropowego, kinematycznego itd.) i przystosowany do modelowania anizotropii związanej z procesem obciążania. W modelu tym stosuje się zmienną wzmocnienia, a więc zmienną dodatkową w porównaniu z teoriami wzmocnienia kinematycznego i izotropowego. Szczególną własność takiego modelu wykorzystano w doświadczeniu przeprowadzonym na próbce ze stopu aluminium poddanej złożonym zmianom i małym odkształceniom.

Среди теорий упруго-пластичности и упруго-вязкоупругости, описывающих состояния малых и конечных деформаций, предлагается в работе модель упрочнения, составляющая обобщение нескольких существующих моделей (изотропной, кинематической и т.д.) и приспособленная к моделированию анизотропии, связанной с процессом нагружения. В этой модели применяется переменная упрочнения, значит дополнительная переменная по сравнению с теориями кинематического и изотропного упрочнения. Особенное свойство такой модели использовано в эксперименте, проведенном на образце из сплава алюминия, подвергнутого сложным изменениям и малым деформациям.

1. Introduction

MODELIZING the evolution of the elastic domain with an anisotropic effect is an important problem to solve. The part it plays in the shaping process regulates the instability conditions. It is therefore easy to understand the numerous attempts to solve this problem [1, 2, 4, 5]. Pre-strains have been taken into account by generalizing the Mises criterion. The hardening effects — kinematic-isotropic as well as the criterion rotation have been carefully described, on the contrary the distortion effect has not.

PHILLIPS [6] tackled the problem with his proposal for a model including the introduction of a graded distortion function describing the threshold surface.

However, no evolution law exists for complex loadings. More general modelizations have been proposed, starting from a threshold isotropic function compared to a set of mathematical objects acting as variables in the function. This way of tackling the problem implies the use of representations theories [1, 3, 8]. But finally, if the domain displacements, rotations and dilatations are well described, there is little relevant information on the threshold function distortion. Finally attempts to tackle the problem from a microscopic

point of view using local inclusion mechanisms and following a statistical distribution can lead to macroscopic laws of behaviour [7, 11, 17].

From this analysis along with our research, a study [15] has been undertaken, a part of which consisted in defining the distortion function more precisely. The latter has been modeled by means of a 2nd order tensor of the deviatoric type and has been introduced into the Mises criterion: the tensor reacts on the threshold surface radius and its evolution law is given under a differential form according to \mathbf{X} , a conjugate variable of kinematic hardening. This study stresses the choice of a single variable (or function) which all the tensor components depend on.

2. Thermodynamic formulation

At present, the finite deformation theory offers several variants which all postulate the existence of a specific Euclidean space E in which the material behaviour is determined, thus allowing to encounter conditions usual in small displacements [16]. Hence the different variables can be determined by means of E tensorial space elements ($T(E)$), particularly the space of E symmetrical endomorphisms is denoted by $\mathcal{L}_s(E) \subset T'_1(E)$

$$(2.1) \quad \begin{aligned} \mathbf{E}_e \in \mathcal{L}_s(E): & \quad \text{elastic strain measure,} \\ \mathbf{P} \in \mathcal{L}_s(E): & \quad \text{stress measure,} \\ \mathbf{D} \in \mathcal{L}_s(E): & \quad \text{strain rate measure.} \end{aligned}$$

The latter is a dual variable of \mathbf{P} , in the sense that the specific or volumic power of cohesion forces is such that:

$$(2.2) \quad P_c = -\langle \mathbf{P}, \mathbf{D} \rangle = -\text{Tr}(\mathbf{P} \cdot \mathbf{D})$$

can be divided into two parts: a plastic part and an elastic part

$$(2.3) \quad \mathbf{D} = \dot{\mathbf{E}}_e + \mathbf{D}_p.$$

Space E depends on the theory considered. The representation can be either of a pure Lagrangean type as in theories in which the loading trihedron is controlled by the evolution of internal parameters, or of a more Eulerian type in the Eulerian approach (in special coordinates related or not to the configuration where the elastic part is not taken into account or finally as in [12], of a Lagrangian type in a corotational space). In the case of small displacements, everything has to be written in the only Euclidean space E , \mathbf{P} is Cauchy endomorphism, \mathbf{D} is the derivative $\dot{\mathbf{E}}$ of strain endomorphism \mathbf{E} . In finite displacements, the definition of \mathbf{E}_e , \mathbf{P} , \mathbf{D} , \mathbf{D}_p , thus the relations between them, depend on the theory considered, and particularly on the choice of E .

We have chosen not to take the variety of theories into account, but to work within the frame of any that one of these theories using the Euclidean space E . Let us suppose that the thermodynamic potential which is similar to the free energy ψ (itself a state function), for an isothermal process, depends only on \mathbf{E}_e and a pair of hardening variables $(p, \boldsymbol{\alpha})$

$$(2.4) \quad \psi = f(\mathbf{E}_e, p, \boldsymbol{\alpha}),$$

(p, α) . p and α are two hardening variables: p is a positive scalar associated with the average isotropic hardening, and α is a deviatoric tensor:

$$(2.5) \quad \alpha \in \mathcal{L}_s^D(E) = \{u \in \mathcal{L}_s(E) \mid \text{Tr}(u) = 0\}$$

associated with a translation of the elastic domain as a whole (kinematic hardening). Traditionally, this potential is supposed to be uncoupled compared to its three variables

$$(2.6) \quad \psi = \phi_1(\mathbf{E}_e) + \phi_2(p) + \phi_3(\alpha).$$

Dissipation is

$$(2.7) \quad \mathcal{D} = \text{Tr}(\mathbf{P}\mathbf{D}) - \phi'_i(\mathbf{E}_e) \cdot \dot{\mathbf{E}}_e - R\dot{p} - \mathbf{X}\dot{\alpha},$$

where R and \mathbf{X} are thermodynamic forces connected to the variables p and α . For example, according to the choice of ϕ_2 and ϕ_3 , the evolution laws of $R = R_0 + Kp^n$ or $R_0 + K\log(np + 1)$ and $\mathbf{X} = \mu\alpha$ with R_0, K, n, μ are material-dependent parameters.

In the elastic process, the dissipation is equal to zero ($\mathbf{D}_p = 0$). Then we have the classical relation

$$(2.8) \quad \mathbf{P} = \phi'_i(\mathbf{E}_e)$$

the expression of dissipation is

$$(2.9) \quad \mathcal{D} = \text{Tr}(\mathbf{P} \cdot \mathbf{D}_p) + R(-\dot{p}) + \mathbf{X}(-\dot{\alpha}).$$

The proposed model uses a supplementary hardening variable. The \mathbf{R} -valued application is defined on a unit sphere of $\mathcal{L}_s^D(E)$ such that

$$(2.10) \quad r: U_s^D(E) \rightarrow \mathbf{R}$$

with

$$U_s^D(E) = \{u \in \mathcal{L}_s^D(E); \|u\| = \text{Tr}(u^2)^{1/2} = 1\}$$

which are to be taken into account when defining the elastic C_e . The latter is defined by its cylindrical equations (\mathbf{s}, I_E) :

$$(2.11) \quad C_e = \{\mathbf{s} \in \mathcal{L}_s(E), f(\mathbf{s}, \mathbf{X}, R, r) = \|\mathbf{s}_D - \mathbf{X}\| - \rho \leq 0\}$$

with

$$\mathbf{s}_D: \text{deviatoric part of } \mathbf{s}; \mathbf{u} = \frac{\mathbf{s}_D - \mathbf{X}}{\|\mathbf{s}_D - \mathbf{X}\|}, \quad \rho = R + h(\mathbf{X}) + r(\mathbf{u}).$$

Thus the variable r is used to modulate the average hardening isotropic part R , using $r(\mathbf{u})$ in every direction of the deviatoric stress space. It is not taken into account in free energy ψ , thus does not play a direct role in dissipation.

Consequently, the dual scalar variables p and R can be considered to be the characteristics of the hardening energy associated to the elastic domain expansion around \mathbf{X} ; thus R and \mathbf{X} are precisely determined in physical terms.

The laws governing the evolution of the variables $\varkappa = (\mathbf{D}_p, \dot{\alpha}, \dot{p})$ are postulated to be normal laws associated with a pseudopotential of dissipation $\Phi(y)$ with $y = (\mathbf{P}, \mathbf{X}, R)$ such that $\varkappa \in \partial\Phi(y)$.

Let us consider that, for every r :

$$(2.12) \quad \bar{C}_e(r) = \{y = (\mathbf{s}, \mathbf{X}, R), z = f(\mathbf{s}, \mathbf{X}, R, r) \leq 0\}.$$

We choose as the pseudo-potential function Φ indicator of the convex $\bar{C}_e(r)$ in elastoplastics and, in elasto-viscoplastics, a function of the following form:

$$\Phi(y) = g(z),$$

where g is a function of \mathbf{R} in \mathbf{R} , convex, positive, equal to zero if $z \leq 0$. The flow law can be thus written, with \mathbf{u}_σ as the unit vector of loading \mathbf{P}

$$(2.13) \quad \begin{aligned} \mathbf{D}_p &= \lambda f'_1(\mathbf{P}, \mathbf{X}, R, r) = \lambda \left[\mathbf{u}_\sigma - \frac{\partial r(u_\sigma)}{\partial \mathbf{P}} \Big|_{r, \mathbf{X}} \right], \\ \dot{\boldsymbol{\alpha}} &= \lambda f'_2(\mathbf{P}, \mathbf{X}, R, r) = \lambda \left[\mathbf{u}_\sigma - \frac{\partial r(u_\sigma)}{\partial \mathbf{X}} \Big|_{r, \sigma_\sigma} - h'(\mathbf{X}) \right], \\ \dot{p} &= \lambda f'_3(\mathbf{P}, \mathbf{X}, R) = \lambda. \end{aligned}$$

λ is equal to $g'(z)$ in elasto-viscoplastics and a positive scalar in elasto-plastics.

Finally, the speed evolution law for the supplementary hardening variable r is postulated as:

$$(2.14) \quad \dot{r} = \|\dot{\mathbf{X}}\| \cdot l(\mathbf{B}, \mathbf{X}, R, r).$$

More precisely, r is defined in terms of its value in each \mathbf{u}

$$(2.15) \quad \forall \mathbf{u} \in U_s^D(E) \dot{r}(\mathbf{u}) = \|\dot{\mathbf{X}}\| \cdot l(\mathbf{P}, \mathbf{X}, R, r(\mathbf{u})).$$

Along the evolution, $\bar{C}_e(r)$, initially chosen as convex, will remain convex for a certain period of time by continuity, but obviously the evolution of r must be carefully looked after from that point of view. In case of non-convexity, other means must be put into action, such as "convexification" of the result field. The type of anisotropy thus described is obviously determined by the choice of l . For $l = 0$ and $h(\mathbf{x})$ (quadratic function), the classical models are encountered with isotropic and kinematic hardening [14, 13].

3. Model-experiment correspondences

To analyse the first experiments carried out in the L. M. T. Cachan, the model used is non standard, of the type MARQUIS [9], CHABOCHE [10] with modifications.

We have considered $h = 0$. The flow laws

$$(3.1) \quad \begin{aligned} \mathbf{D}_p &= \lambda \left[\mathbf{u}_\sigma - \frac{\partial r(u_\sigma)}{\partial \mathbf{P}} \Big|_{r, \mathbf{X}} \right], \\ \dot{\boldsymbol{\alpha}} &= \lambda \left[\mathbf{u}_\sigma + \frac{\partial r(u_\sigma)}{\partial \mathbf{X}} \Big|_{r, \sigma} - \phi(p) \cdot \mathbf{X} \right], \\ \dot{p} &= \lambda \end{aligned}$$

with

$$\phi(p) = \frac{1}{C_0 + C_\infty(1 - e^{-\omega p})}$$

a first modelization of the function l : with a single parameter a such that

$$(3.2) \quad l(u; \mathbf{P}_x, R, r(u)) = a \frac{\|x\|^2}{\|\mathbf{P}_D - \mathbf{x}\|^2} \{A + 3/2(1-A)\} \{1 - \text{Tr}(u \cdot u_\sigma)\} \\ \times \{1 - e^{2\text{Tr}(u \cdot u_x)}\} \text{Tr}(u \cdot u_x)$$

with

$$A = \frac{\text{Tr}(\mathbf{P}_\sigma)}{\|\mathbf{P}\|} \quad \text{and} \quad u_x = \frac{\mathbf{x}}{\|\mathbf{X}\|}.$$

We have compared the model to experimental determinations of the elastic boundary evolution under complex loadings [15] (carried out on an aluminium alloy type 2024) and the comparison is valid.

The elastic boundary seems to be well described qualitatively. The nature and magnitude of the loading play an important role, especially in complex loading. To modelize these effects, we have introduced the norm $\|\mathbf{P}_D - \mathbf{X}\|$ associated to magnitude into the function

Table 1.

Hardening	$\frac{K}{da} \text{ Nmm}^2$	n	$\frac{R_0}{da} \text{ Nmm}^2$	$\frac{N}{da} \text{ Nmm}^2$	C_0	C_∞	w	a
Isotropic	80	0.3	164					
Kinematic				80.000	49	160	20	
Distorsion								0.7

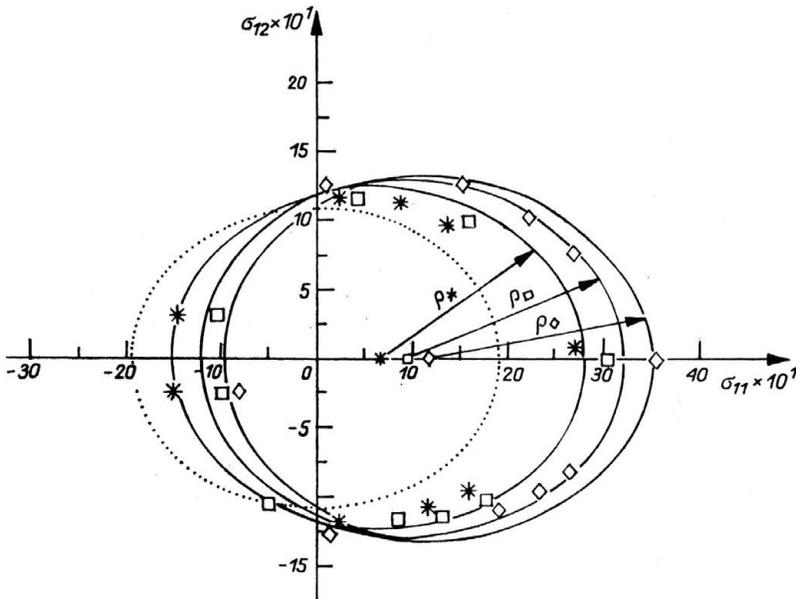


FIG. 1.

1. The quantity A describes the nature of the loading by making a distinction between hydrostatic and non-hydrostatic effects. In the longer run, not only could we integrate induced anisotropy into " l " but also those effects due to the texture.

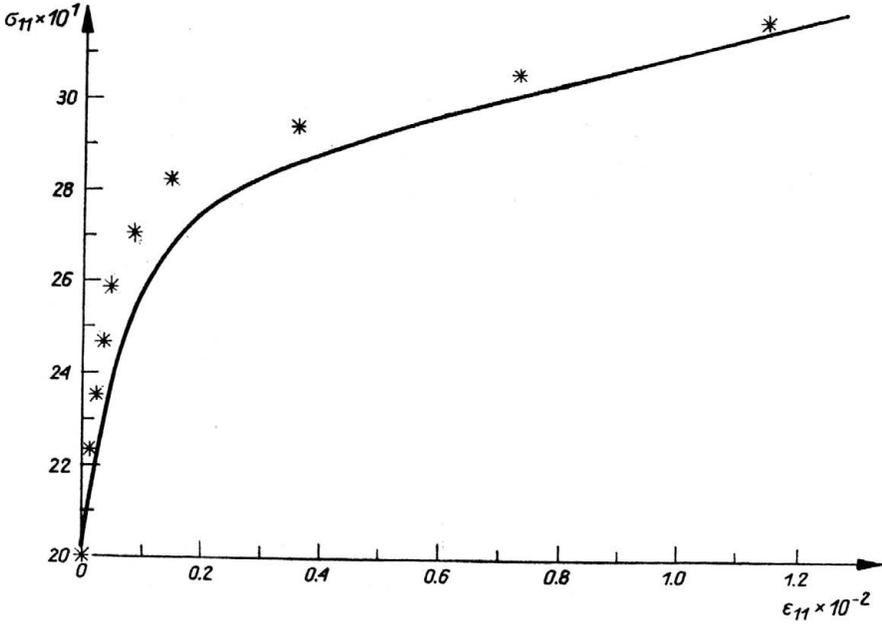


FIG. 2.

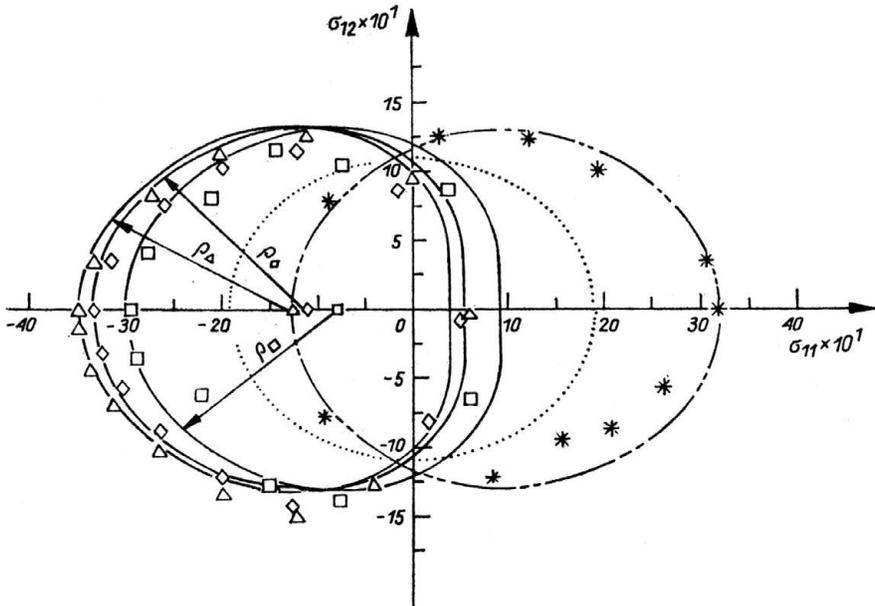


FIG. 3.

Following a minimization, we find the parameters which modelize the material. They have been disposed according to their domain in the Table 1. In the following figures, the curves continuous lines have been calculated using the parameters above, whereas experimental values have been drawn using various geometrical figures. The loading is

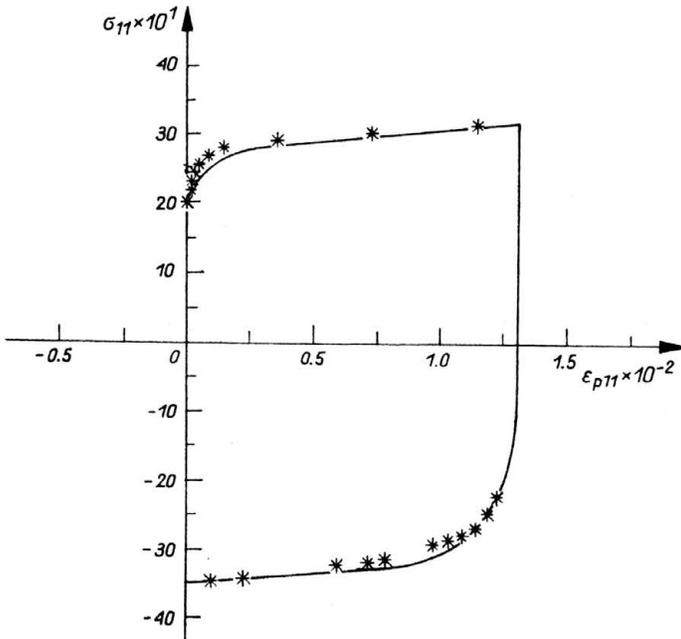


FIG. 4.

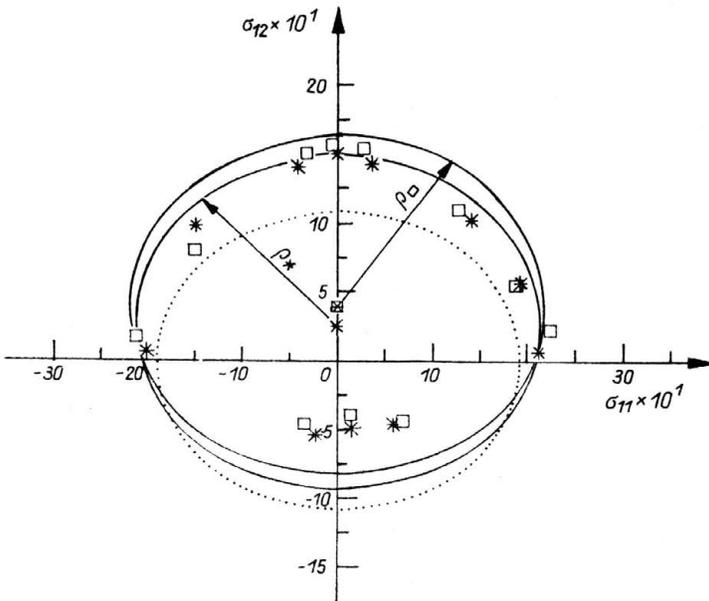


FIG. 5.

defined in terms of plane stress with two variables (σ_{11} : tension and compression, σ_{12} : torsion).

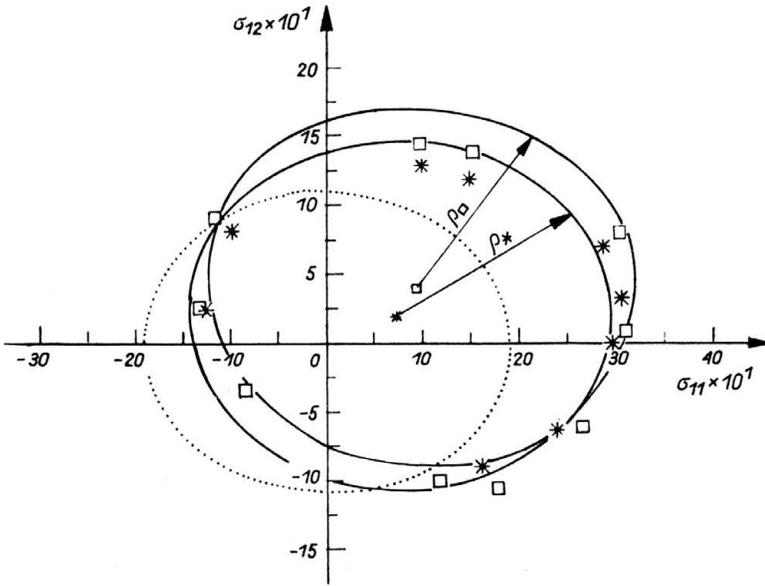


FIG. 6.

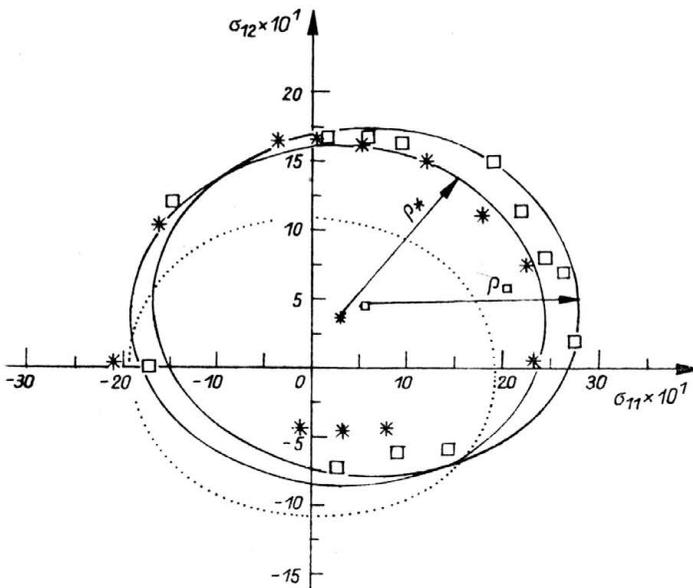


FIG. 7.

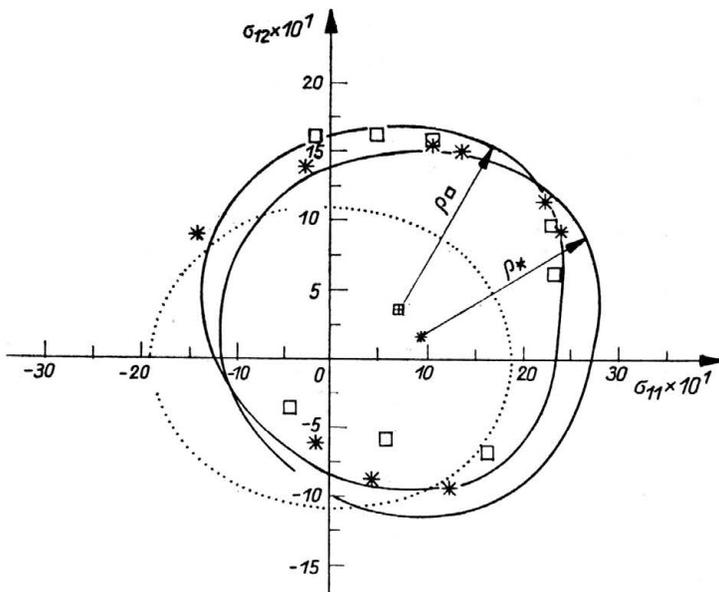


FIG. 8.

Table 2.

Figure	Loading type	σ_{11}	σ_{12}	X_{11}	X_{12}	R
no. 1	Tension	280		63.7		176.6
		320		94.1		184.4
		350		119.2		188.4
no. 3	Tension ↓ Compression	317		94		184.3
		-300		-79		185.2
		-350		-112		187.4
		-355		-124.8		189.2
no. 5	Torsion		150		27.2	173.7
			165		39.2	177.8
no. 6	Tension ↓ Torsion	275				
			75	74.8	19.6	181
no. 7	Torsion ↓ Tension		150			
		120		26.8	36.5	178.2
no. 8	Tension ↓ elastic tension unloading	350				
		120				
	Torsion		150	91.3	17.8	188.6
			165	68.2	35.4	188.7

Three categories in loading:

simple: tension (Fig. 1), torsion (Fig. 5),

complex: tension \rightarrow compression (Fig. 3),

tension \rightarrow torsion (Fig. 6),

torsion \rightarrow tension (Fig. 7),

complex with partial elastic unloading:

tension \rightarrow elastic unloading tension \rightarrow torsion (Fig. 8).

Figures 2 and 4 are, respectively, behaviour laws for tension and tension-compression experiments. The computed results are given in the Table 2.

4. Conclusion

The model we have presented includes a single distortion parameter to describe all distortion effects. Despite the fact that there is only one parameter, all distortion effects are correctly represented for the material studied.

Before improving the model itself, it would be more appropriate to standardize it as defined in Sect. 2. Moreover, further experiments on different types of material should be carried out and the results analysed so as to extend the domain of the model. The next step should be to extend the model for the plastic deformation process going as far as fracture and the effects of damage in the elastic domain.

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