

## Film flow of an inviscid liquid(\*)

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THE PAPER is concerned with the steady flow of an inviscid liquid in a film which is formed when "rain" is falling onto a solid bottom. At first an exact solution of the problem is constructed. Then, an approximation method is derived and is finally tested by the exact solution.

Praca dotyczy stacjonarnego przepływu nielepkiej cieczy w cienkiej warstwie tworzącej się na cieple sztywnym podczas padania „deszczu”. Podano ścisłe rozwiązanie zagadnienia a także rozwiązanie przybliżone. W zakończeniu pracy dokonano porównania obu rozwiązań.

Работа касается стационарного течения невязкой жидкости в тонком слое, образующемся на жестком теле во время осадков „дождя”. Дается точное решение задачи, а также приближенное решение. В заключении работы проведено сравнение обоих решений.

### 1. Introduction

THIS PAPER deals with the following problem: Determine the flow in the film which is formed when "rain" is falling onto an impervious bottom. First of all one thinks of films caused by natural rain, of course. But those films are so thin that the viscosity plays an important role (the film thickness is of the same order as the boundary layer thickness), whereas we shall neglect the viscosity. There are two arguments for this simplifying assumption: Firstly, there are cases in which the films are much thicker than the usual rain films; one can think, for example, of artificial sprinkling. Secondly, investigations for an ideal liquid should be helpful for the much more difficult viscous theory which leads to very complicated calculations if one has an arbitrary bottom shape or if one wants to simulate the discrete "rain" drops. (Compare the final remarks to chapter 4).

The problem of viscous film flow has been considered by several authors (see BECKER 1976 and the bibliographical data there). The inviscid case was investigated at first by Becker (see BECKER 1975). (In the hydraulic approximation he derived an integrodifferential equation for the film thickness).

### 2. Assumptions and basic equations

We use a left-handed Cartesian coordinate system  $(x, y)$  with the  $y$ -axis in the direction of the gravity acceleration  $\mathbf{g}$ , and we consider a plane steady flow of an inviscid liquid with the constant density  $\rho_0$  in the film domain  $D$ .  $D$  is bounded by the bottom  $b(x)$  and by the film surface  $s(x)$ , which has the tangential vector  $\mathbf{t}_s$ , the inner normal vector  $\mathbf{n}_s$ , and the angle of inclination  $\alpha$  (see Fig. 1).

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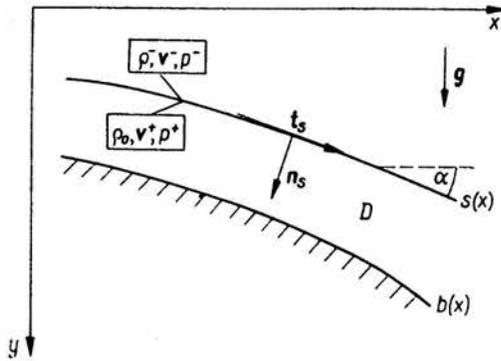


FIG. 1. Notations.

As usual, we replace the "rain" by a continuously distributed fluid. In addition we assume that this continuous "rain" flow is steady. Hence at the surface  $s(x)$  the time-independent outer limits of the density, the velocity, and the pressure:  $\rho^-$ ,  $v^-$ ,  $p^-$  can be defined. Then the inner limits  $v^+$  and  $p^+$  ( $\rho^+ = \rho_0 = \text{const}$  is given) result from the jump conditions for mass

$$(2.1) \quad [[\rho v]] \cdot n_s = 0$$

and momentum,

$$(2.2) \quad [[\rho(v \cdot n_s)v + p n_s]] = 0.$$

(The brackets denote the differences between the inner and outer limits).

The flow in the film domain  $D$  is described by the continuity equation

$$(2.3) \quad \text{div } v = 0$$

and the Euler equations.

$$(2.4) \quad v \times \text{curl } v = \text{grad} \left( \frac{v^2}{2} + \frac{p}{\rho_0} - gy \right).$$

These equations lead to following differential equation for the stream function  $\psi$ :

$$(2.5) \quad \Delta \psi = \frac{d}{d\psi} B(\psi)$$

with

$$(2.6) \quad B = \frac{v^2}{2} + \frac{p}{\rho_0} - gy.$$

We have the following boundary conditions:

The impervious bottom  $b(x)$  is a streamline, i.e.

$$(2.7) \quad \text{at } b(x): \psi = \text{const} (= 0).$$

The boundary value of  $\psi$  at the film surface, which we denote by  $\psi^+$ , is a function  $f(x)$  determined by the normal velocity

$$(2.8) \quad \text{at } s(x): \psi^+ = - \int_{x_0}^x v \cdot n_s \frac{dx}{\cos \alpha(x)} = f(x).$$

In addition we have

$$(2.8) \quad \text{at } s(x): \left( \frac{\partial \psi}{\partial n} \right)^+ = \mathbf{v}^+ \cdot \mathbf{t}_s.$$

In order to solve the differential equation (2.5) with the boundary conditions (2.7) and (2.8) we must know the function  $B(\psi)$ . Its inner limit is given by the jump conditions

$$(2.9) \quad B^+(x) = \frac{\mathbf{v}^{+2}(x)}{2} + \frac{p^+(x)}{\rho_0} - g s(x).$$

To construct  $B(\psi)$  in  $D$  we make two assumptions:

**Assumption 1.** Liquid is added at the whole film surface.

**Assumption 2.** All the liquid in the film is added at the surface.

From Assumption 1 it follows that  $f(x)$  is a monotonously decreasing function. So it can be inverted in the following way:

$$(2.10) \quad x = f^{-1}(\psi^+).$$

Inserting this into  $B^+(x)$  leads to a function  $B(\psi^+)$  defined by

$$(2.11) \quad B(\psi^+) = B^+[f^{-1}(\psi^+)]$$

and it is trivial to omit the  $+ -$  signs to get  $B(\psi)$ :

$$(2.12) \quad B(\psi) = B^+[f^{-1}(\psi)].$$

Assumption 2 implies that this equation holds in the whole domain  $D$ .

The function  $B(\psi)$  depends on the shape of the film surface because it is constructed with the aid of the function  $B^+(x)$ . Therefore, the right-hand side of Eq. (2.5) is not known a priori, consequently the film flow for the given bottom  $b(x)$  can only be determined by successive approximation. This suggests to look at first for simple solutions of the inverse problem, i.e. the problem to construct the flow for the given surface  $s(x)$ . Sometimes this method is applied to the usual free surface flows, too (see WEHAUSEN and LAITONE 1960, pp. 736).

### 3. A solution of the inverse problem

To get a simple solution of the inverse problem we choose the film surface such that  $B^+(x)$  is constant. Then  $B(\psi)$  is constant (see Eq. (2.12)) and  $\Delta\psi$  is zero (see Eq. (2.5)):

$$(3.1) \quad B^+(x) = \text{const} \Leftrightarrow B(\psi) = \text{const} \Leftrightarrow \Delta\psi = 0 \text{ in } D$$

i.e. the corresponding film flow is a potential flow.

**REMARK.** That this flow is free of vorticity is not inconsistent with the fact that strong vorticities are supplied to the real flow by the diving fluid drops. This is so since replacing the "rain" by a continuum means taking an average, and so  $B(\psi) = \text{const}$  only signifies that the vorticity is zero on an average.

We use the simplest "rain" model:

$$(3.2) \quad \begin{aligned} \mathbf{v}^- &= (0, V), & V &= \text{const}, \\ \varrho^- &= \varepsilon \varrho_0, & \varepsilon &= \text{const}, & 0 < \varepsilon < 1, \\ p^- &= 0. \end{aligned}$$

For this "rain" the jump conditions lead to the following inner limits of the normal velocity, the tangential velocity and the pressure:

$$(3.3) \quad \begin{aligned} \mathbf{v} \cdot \mathbf{n}_s &= \varepsilon V \cos \alpha, \\ \mathbf{v} \cdot \mathbf{t}_s &= V \sin \alpha, \\ \frac{p^+}{\varrho_0} &= \varepsilon(1-\varepsilon)V^2 \cos^2 \alpha. \end{aligned}$$

Inserting these results into Eq. (2.9), we get

$$(3.4) \quad B^+(x) = \frac{1}{2} V^2 - g s(x) - \frac{(1-\varepsilon)^2}{2} V^2 \frac{1}{1+s'^2(x)}.$$

We are looking for solutions with  $B = \text{const}$ .

This constant can be chosen such that  $s = 0$  when  $s' = 0$ . The resulting differential equation has as solution the cycloid

$$(3.5) \quad z_s(\alpha) = \frac{R}{4} [2\alpha + \sin 2\alpha + i(1 - \cos 2\alpha)]$$

with

$$z_s = x_s + iy_s, \quad R = (1-\varepsilon)^2 \frac{V^2}{g}, \quad s = y_s(\alpha), \quad x = x_s(\alpha).$$

The boundary values of the velocity potential  $\varphi$  (whose existence is guaranteed by Eq. (3.1)) and of the streamfunction  $\psi$

$$(3.6) \quad \begin{aligned} \varphi^+(x) &= \int_0^x \mathbf{v}^+ \cdot \mathbf{t}_s \frac{dx}{\cos \alpha} = V s(x), \\ \psi^+(x) &= - \int_0^x \mathbf{v}^+ \cdot \mathbf{n}_s \frac{dx}{\cos \alpha} = -\varepsilon V x \end{aligned}$$

fix the boundary value  $F^+$  of the complex potential  $F = \varphi + i\psi$ .

By Eq. (3.5) it can be transformed into the following function of  $\alpha$ :

$$(3.7) \quad F^+(\alpha) = V \frac{R}{4} [1 - \cos 2\alpha - i\varepsilon(2\alpha + \sin 2\alpha)].$$

To determine the film flow belonging to  $z_s(\alpha)$ , a function  $F(z)$  has to be constructed, regular in  $D$ , which takes the values prescribed by Eq. (3.7) on  $z_s(\alpha)$ . It is known from the theory of functions (see e.g. BEHNKE and SOMMER 1965, p. 140) that this problem has one solution at the most, and because both  $z_s(\alpha)$  and  $F^+(\alpha)$  are analytic between  $-\frac{\pi}{2}$

and  $+\frac{\pi}{2}$ , this solution is obtained in parameter form by replacing the real parameter  $\alpha$  in Eqs. (3.5) and (3.7) by the complex parameter  $\gamma = \alpha + i\beta$ . So we find;

$$(3.8) \quad \begin{aligned} z(\gamma) &= \frac{R}{4} [2\gamma + i(1 - e^{i2\gamma})], \\ F(\gamma) &= V \frac{R}{4} [1 - \cos 2\gamma - i\epsilon(2\gamma + \sin 2\gamma)]. \end{aligned}$$

Equation (3.8)<sub>2</sub> can be transformed as follows:

$$(3.9) \quad \begin{aligned} F(\gamma) &= V \frac{R}{4} \{ (1 - \epsilon)(1 - \cos 2\gamma) - \epsilon i [2\gamma + i + \sin 2\gamma - i \cos 2\gamma] \} \\ &= (1 - \epsilon)V \frac{R}{4} (1 - \cos 2\gamma) - \epsilon i V z(\gamma) = (1 - \epsilon)F_0 + \epsilon F_1. \end{aligned}$$

Thus we can interpret  $F(\gamma)$  as the superposition, weighted by  $\epsilon$ , of the flow  $F_0 = V \frac{R}{4} (1 - \cos 2\gamma)$  with a parallel flow in the  $y$ -direction. (For the meaning of  $F_0$  see the final remark to this chapter).

By separating  $z(\gamma)$  and  $F(\gamma)$  into their real and imaginary parts, we get

$$(3.10) \quad \begin{aligned} z &= \frac{R}{4} [\sin 2\alpha e^{-2\beta} + 2\alpha] + i \frac{R}{4} [1 - \cos 2\alpha e^{-2\beta} + 2\beta], \\ F &= V \frac{R}{4} [1 - \cos 2\alpha (\text{ch} 2\beta - \epsilon \text{sh} 2\beta) + \epsilon 2\beta] + i V \frac{R}{4} [\sin 2\alpha (\text{sh} 2\beta - \epsilon \text{ch} 2\beta) - \epsilon 2\alpha] \end{aligned}$$

with  $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$ ,  $\beta \geq 0$ .

According to Eq. (3.10)<sub>2</sub> the streamlines of the flow satisfy the equation

$$(3.11) \quad \psi = V \frac{R}{4} [\sin 2\alpha (\text{sh} 2\beta - \epsilon \text{ch} 2\beta) - \epsilon 2\alpha] = \text{const.}$$

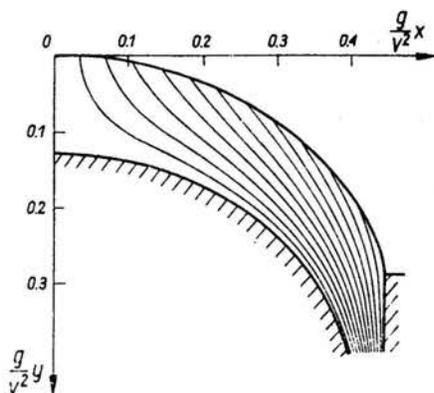


FIG. 2. Potential film flow for a liquid "raining" with constant velocity  $V$ ,  $\epsilon = \varrho^-/\varrho_0 = \frac{1}{4}$ .

For  $\alpha \neq 0$  this defines a function  $\beta(\psi; \alpha)$ . Inserting it into Eq. (3.10)<sub>1</sub> we obtain the streamlines in  $D$  as function  $z(\psi; \alpha)$ . (For  $\psi = 0$ ,  $\alpha = 0$ , and  $\beta = \frac{1}{2} \ln \frac{1+\varepsilon}{1-\varepsilon}$  the flow has a stagnation point). Some streamlines are plotted for  $\varepsilon = \frac{1}{4}$  and  $\varepsilon = \frac{1}{16}$  in Figs. 2 and 3.

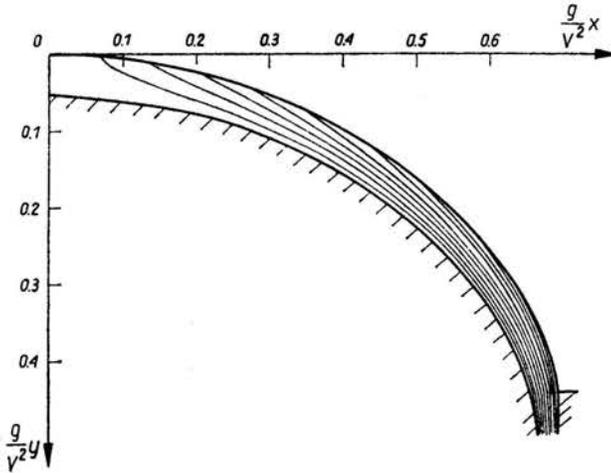


FIG. 3. Potential film flow for a liquid "raining" with constant velocity  $V$ ,  $\varepsilon = \varrho^- / \varrho_0 = \frac{1}{16}$ .

(The straight half line  $x = \frac{R}{4} \pi$ ,  $y \geq \frac{R}{2}$  cannot be conceived as a part of the film surface because the jump condition (2.1) and (2.2) are violated here). We supposed that  $\varepsilon$  lies between 0 and 1. In the limit  $\varepsilon \rightarrow 1$  the film surface shrinks up to a point (see Eq. (3.5):  $R \rightarrow 0$ ). In the limit  $\varepsilon \rightarrow 0$  the normal velocity  $\mathbf{v} \cdot \mathbf{n}_s = \varepsilon V \cos \alpha$  vanishes, i.e. the surface becomes a streamline, and  $B^+(x) = \text{const}$  implies that this streamline is a free surface. So the flow  $F_0 = V \frac{R}{4} (1 - \cos 2\gamma)$ , which is the limit of  $F$  when  $\varepsilon$  goes to zero, is a flow of a heavy fluid with a free surface. This flow is known all along. Apparently, it was first investigated by N. E. Zhukowskii in 1891 (see GUREVICH 1965, p. 543 and the bibliographical date there, or GILBARG, 1960, p. 351).

With the help of the exact solution constructed in this chapter we can test the following approximation method.

#### 4. A successive approximation method

The governing differential equation (2.5)  $\Delta \psi = \frac{dB}{d\psi}$  can be transformed into a system of three equations by reintroducing the velocity field  $(u, v)$ . These three equations are:

$$(4.1) \quad \begin{aligned} u &= \frac{\partial \psi}{\partial y}, \\ v &= -\frac{\partial \psi}{\partial x}, \\ \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} &= \frac{dB}{d\psi} [\psi(x, y)]. \end{aligned}$$

In the following we assume  $u > 0$  in the considered domain. Hence Eqs. (4.1)<sub>1</sub> and (4.1)<sub>2</sub> describe a coordinate transformation

$$(4.2) \quad (x, y) \rightarrow (x, \psi).$$

The inverse equations corresponding to Eqs. (4.1)<sub>1</sub> and (4.1)<sub>2</sub> are

$$(4.3) \quad \begin{aligned} \frac{1}{u} &= \frac{\partial y}{\partial \psi}, \\ \frac{v}{u} &= \frac{\partial y}{\partial x}, \end{aligned}$$

and in the new coordinate system Eq. (4.1)<sub>3</sub> takes the form

$$(4.3)_3 \quad u \frac{\partial u}{\partial \psi} + v \frac{\partial v}{\partial \psi} - \frac{\partial v}{\partial x} = \frac{dB}{d\psi}.$$

The boundary conditions for the system (4.3) are

$$(4.4) \quad \begin{aligned} \text{at } \psi = 0: y &= b(x); \quad \left( \frac{v}{u} = b'(x) \right), \\ \text{at } \psi = \psi^+(x): y &= S(x), \quad u = u^+(x), \quad v = v^+(x), \quad B = B^+(x). \end{aligned}$$

Integrating the system (4.3) over  $\psi$  and satisfying the boundary conditions (4.4), we get

$$(4.5) \quad \begin{aligned} s(x) &= b(x) + \int_0^{\psi^+(x)} \frac{d\tau}{u(x, \tau)}, \\ \frac{v}{u}(x, \psi) &= b'(x) + \int_0^{\psi} \frac{\partial}{\partial x} \left[ \frac{1}{u(x, \tau)} \right] d\tau, \end{aligned}$$

$$u^2(x, \psi) + v^2(x, \psi) = u^{+2}(x) + v^{+2}(x) + 2[B(\psi) - B^+(x)] + 2 \int_{\psi^+(x)}^{\psi} \frac{\partial}{\partial x} v(x, \tau) d\tau.$$

Now we confine the consideration to the "rain" model (3.2) (the following can be modified for more general "rains"), and we conceive all the quantities to be made dimensionless by the rain velocity  $V$  and a characteristic length  $R$  (which is for example an average radius of curvature of the impervious bottom). Hence the dimensionless streamfunction has the boundary value

$$(4.6) \quad \psi^+(x) = -\varepsilon x.$$

This relation suggests to introduce a new coordinate  $\hat{\psi}$  by

$$(4.7) \quad \psi = -\varepsilon \hat{\psi}.$$

In the coordinate system  $(x, \hat{\psi})$  and for the special "rain" model (3.2) Eqs. (4.5) take the following dimensionless form:

$$(4.8) \quad \begin{aligned} s(x) &= b(x) - \varepsilon \int_0^x \frac{d\hat{\tau}}{u(x, \hat{\tau})}, \\ \frac{v}{u}(x, \hat{\psi}) &= b'(x) - \varepsilon \int_0^{\hat{\psi}} \frac{\partial}{\partial x} \left[ \frac{1}{u(x, \hat{\tau})} \right] d\hat{\tau}, \\ u^2(x, \hat{\psi}) + v^2(x, \hat{\psi}) &= (1 - \varepsilon)^2 \sin^2 \alpha(\hat{\psi}) + \frac{2}{F_r^2} [s(x) - s(\hat{\psi})] \\ &\quad + 2\varepsilon(1 - \varepsilon) \sin^2 \alpha(x) + \varepsilon^2 + 2\varepsilon \int_{\hat{\psi}}^x \frac{\partial}{\partial x} v(x, \hat{\tau}) d\hat{\tau}, \end{aligned}$$

with

$$F_r^2 = \frac{V^2}{gR}, \quad \operatorname{tg} \alpha = \frac{ds}{dx},$$

$(u^+(x)$  and  $v^+(x)$  result from Eq. (3.3) and both  $B^+(x)$  and  $B(\hat{\psi})$  from Eq. (3.4). Note that  $B(\hat{\psi}) = B^+(\hat{\psi})$  because  $\hat{\psi}^+(x) = x$ .)

Now we assume that the solutions  $u(x, \hat{\psi})$ ,  $v(x, \hat{\psi})$ ,  $s(x)$  of the system (4.8) can be expanded into powers of the small parameter  $\varepsilon$ :

$$(4.9) \quad \begin{aligned} u &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots, \\ v &= v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \dots, \\ s &= s_0 + \varepsilon s_1 + \varepsilon^2 s_2 + \dots \end{aligned}$$

Inserting these expansions into Eqs. (4.8), we get the following first approximations:

$$(4.10) \quad \begin{aligned} s_0(x) &= b(x), \\ v_0(x, \hat{\psi}) &= b'(x) u_0(x, \hat{\psi}), \\ u_0(x, \hat{\psi}) &= \frac{1}{\sqrt{1 + b'^2(x)}} \sqrt{\frac{b'^2(\hat{\psi})}{1 + b'^2(\hat{\psi})} + \frac{2}{F_r^2} [b(x) - b(\hat{\psi})]}. \end{aligned}$$

The first nontrivial term in the expansion of  $s$  is

$$(4.11) \quad s_1(x) = -\sqrt{1 + b'^2(x)} \int_0^x \frac{d\hat{\tau}}{\sqrt{\frac{b'^2(\hat{\tau})}{1 + b'^2(\hat{\tau})} + \frac{2}{F_r^2} [b(x) - b(\hat{\tau})]}}.$$

Hence we obtain a first approximation for the film height  $b - s$  simply by a quadrature.

#### REMARKS:

1) In some cases it may be advantageous not to consider the vertical film height  $b - s$ , but the film thickness  $d$  normal to the bottom  $b$ . It is easy to show that

$$(4.12) \quad d_1 = -\frac{1}{\sqrt{1 + b'^2(x)}} s_1.$$

2) The above formulae can easily be generalized to the case that  $\varepsilon$  is given by

$$(4.13) \quad \varepsilon = \varepsilon_0 g(x), \quad 0 < g(x) (\leq 1), \quad 0 < \varepsilon_0 < 1.$$

Then we have to replace Eq. (4.7) defining  $\hat{\psi}$  by

$$(4.14) \quad \psi = f(\hat{\psi}) = -\varepsilon_0 \int_0^{\hat{\psi}} g(\hat{\tau}) d\hat{\tau}$$

which again leads to  $\hat{\psi}^+(x) = x$ . In that case we obtain

$$(4.15) \quad S_1(x) = -\sqrt{1+b'^2(x)} \int_0^x \frac{g(\hat{\tau}) d\hat{\tau}}{\sqrt{\frac{b'^2(\hat{\tau})}{1+b'^2(\hat{\tau})} + \frac{2}{F_r^2} [b(x)-b(\hat{\tau})]}}$$

3) The integral in Eq. (4.11) tends to infinity for  $x \rightarrow x_0$  if  $b'(x_0) = 0$ . Therefore, the formula (4.11) can be applied immediately only if  $b'(x) > 0$  for  $x > 0$  (resp.  $b'(x) < 0$  for  $x < 0$ ). If that condition is violated, we have to split up the function  $b(x)$  into the following sum for each fixed  $\varepsilon$  (resp.  $\varepsilon_0$ ) under consideration:

$$(4.16) \quad b(x) = b_0(x) + \sum_{\nu=1}^{\eta} \varepsilon^\nu b_\nu(x)$$

with

$$b'_0(x) > 0 \text{ for } x > 0 \quad (\text{resp. } b'_0(x) < 0 \text{ for } x < 0),$$

$$\max |b_\nu(x)| = 0(1), \quad n \leq \infty.$$

Consequently, in Eq. (4.11)  $b(x)$  must be replaced by  $b_0(x)$ . Of course, the expansion (4.16) is not possible for all smooth functions  $b(x)$ . But we should not forget that there are functions  $b(x)$  for which our problem cannot have a steady solution. Such a function is, for example,  $b(x) = -cx^2$  with  $c \geq 0$ .

### 5. Comparison with the solution of Chapter 3

Of course, a strict convergence proof for the expansions of the preceding chapter would be very difficult. But we can test the procedure with the help of the exact solution constructed in Chapter 3. For this purpose we choose as the characteristic length

$$(5.1) \quad R = (1-\varepsilon)^2 \frac{V^2}{g}.$$

Then we obtain for the bottom (see Eqs. (3.10)<sub>1</sub> and (3.11)):

$$(5.2) \quad x_b = \frac{1}{4} (\sin 2\alpha e^{-2\beta} + 2\alpha),$$

$$y_b = \frac{1}{4} (1 - \cos 2\alpha e^{-2\beta} + 2\beta),$$

$$4\psi_b = \sin 2\alpha (\text{sh } 2\beta - \varepsilon \text{ch } 2\beta) - \varepsilon 2\alpha = 0.$$

We define a new parameter  $\vartheta$  by

$$(5.3) \quad \sin 2\vartheta + 2\vartheta = \sin 2\alpha\epsilon^{-2\vartheta} + 2\alpha$$

and expand  $\alpha$  and  $\beta$  into powers of  $\epsilon$ :  $\alpha = \alpha_0 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + \dots$ ,  $\beta = \beta_0 + \epsilon\beta_1 + \epsilon^2\beta_2 + \dots$ .

Inserting this into Eqs. (5.2)<sub>3</sub> and (5.3) gives  $\alpha_i(\vartheta)$ ,  $\beta_i(\vartheta)$  (especially  $\alpha_0 = \vartheta$ ,  $\beta_0 = 0$ ).

So we get the bottom expanded into powers of  $\epsilon$ :

$$(5.4) \quad \begin{aligned} x_b &= \frac{1}{4} (\sin 2\vartheta + 2\vartheta), \\ y_b &= \frac{1}{4} (1 - \cos 2\vartheta) + \frac{\epsilon}{2} \left( 1 + \frac{2\vartheta}{\sin 2\vartheta} \right) + \frac{\epsilon^2}{4} \left[ 2 \left( 1 + \frac{2\vartheta}{\sin 2\vartheta} \right) \right. \\ &\quad \left. - \frac{1 + 2\cos 2\vartheta}{1 + \cos 2\vartheta} \left( 1 + \frac{2\vartheta}{\sin 2\vartheta} \right)^2 \right] + O(\epsilon^3). \end{aligned}$$

The corresponding film surface (see Eq. (3.5)) is

$$(5.5) \quad \begin{aligned} x_s &= \frac{1}{4} (\sin 2\vartheta + 2\vartheta), \\ y_s &= \frac{1}{4} (1 - \cos 2\vartheta). \end{aligned}$$

In the special case considered here the bottom and the Froude number  $\text{Fr}^2 = \frac{V^2}{gR} = 1/(1-\epsilon)^2$  depend on  $\epsilon$ . Therefore, the formulae (4.10) and (4.11) must be modified a little bit. We start again from Eq. (4.8) and obtain at first from Eq. (4.8)<sub>1</sub>:  $s_0 = b_0$ , i.e.

$$(5.6) \quad y_{s0} = y_{b0}$$

which is in agreement with Eqs. (5.4)<sub>2</sub> and (5.5)<sub>2</sub>. Then Eq. (4.8)<sub>2</sub> gives

$$(5.7) \quad v_0 = b'_0 u_0 = \text{tg } \vartheta u_0$$

and herewith Eq. (4.8)<sub>3</sub> leads to

$$(5.8) \quad u_0 = \frac{1}{2} \sin 2\vartheta.$$

Inserting this into Eq. (4.8)<sub>1</sub> we obtain

$$(5.9) \quad s_1(x) = b_1(x) - \frac{2x}{\sin 2\vartheta(x)}$$

or, in parameter form,

$$(5.10) \quad y_{s1}(\vartheta) = y_{b1}(\vartheta) - \frac{1}{2} \left( 1 + \frac{2\vartheta}{\sin 2\vartheta} \right)$$

which again is in agreement with Eqs. (5.4)<sub>2</sub> and (5.5)<sub>2</sub>. In the next step we obtain from Eqs. (4.8)<sub>2</sub>, (4.8)<sub>3</sub> and (4.8)<sub>1</sub>, respectively,

$$(5.11) \quad v_1(x, \hat{\psi}) = \text{tg } \vartheta(x) u_1(x, \hat{\psi}) + 1 - \frac{\cos 2\vartheta}{1 + \cos 2\vartheta}(x) 4 \frac{x - \hat{\psi}}{\sin 2\vartheta(x)},$$

$$(5.12) \quad u_1(x, \hat{\psi}) = -\frac{1}{2} \sin 2\vartheta(x) + \frac{1 + 2\cos 2\vartheta}{1 + \cos 2\vartheta}(x) 2(x - \hat{\psi}),$$

$$(5.13) \quad s_2(x) = b_2(x) - \frac{2x}{\sin 2\vartheta(x)} + \frac{1 + 2\cos 2\vartheta}{1 + \cos 2\vartheta}(x) \left[ \frac{2x}{\sin 2\vartheta(x)} \right]^2,$$

or, in parameter form,

$$(5.14) \quad y_{s2}(\vartheta) = y_{b2}(\vartheta) - \frac{1}{2} \left( 1 + \frac{2\vartheta}{\sin 2\vartheta} \right) + \frac{1+2\cos 2\vartheta}{1+\cos 2\vartheta} \frac{1}{4} \left( 1 + \frac{2\vartheta}{\sin 2\vartheta} \right)^2$$

which corresponds with Eqs. (5.4)<sub>2</sub> and (5.5)<sub>2</sub>.

The author has also tested the validity of the next approximation. It can be expected that all the further terms of the expansions are correct. But it must be noticed that the exact solution cannot be expanded into powers of  $\varepsilon$  for  $\vartheta$  close to  $\pi/2$  as Eq. (5.4)<sub>2</sub> shows. Hence the procedure must fail there.

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