

Asymptotic method of homogenization of two models of elastic shells

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THIS PAPER deals with homogenization of Koiter's linear shell and the geometrically nonlinear shallow shell. To derive the effective moduli, the method of two scale asymptotic expansions has been employed. In the case of Koiter's shell, the local problems are coupled. They become uncoupled for shallow shells. An example is also given.

W pracy rozpatrzono homogenizację liniowej powłoki Koitera i geometrycznie nieliniowej powłoki małowyniosłej. W celu otrzymania modułów efektywnych zastosowano dwuskalową metodę rozwinięć asymptotycznych. W przypadku powłoki Koitera zagadnienia lokalne są sprzężone. Stają się one rozsprężone dla powłok małowyniosłych. Podano również przykład.

В работе рассмотрена гомогенизация линейной оболочек Койтера и геометрически нелинейной пологой оболочки. С целью получения эффективных модулей применен двухмасштабный метод асимптотических разложений. В случае оболочки Койтера локальные задачи сопряжены. Становятся они распряженными для пологих оболочек. Приведен тоже пример.

1. Introduction

SHELLS of periodically varying stiffness, e.g. shells with ribs, with openings or fibre-reinforced, are often used in engineering practice. The aim of this paper is to provide an effective method for the statical analysis of such structures.

The first part of the paper is devoted to the linear analysis of a Kirchhoff-Love shell (we use Koiter's version of this theory, [9]) being periodic with respect to assumed curvilinear parametrization. Under the assumption that a cell of periodicity has the shape of a shallow shell, using the method of asymptotic expansions [5, 8], formulae will be derived for effective stiffnesses (which are non-constant). Although homogenization methods are widely used for periodic composites (cf. [5, 30]), there is a limited number of papers in which they are applied to plates and arches [1-3, 7, 11-14, 16-19, 21, 22, 26-30, 32, 34, 38, 39]. The problem of homogenization of shells has been dealt with in the papers [18, 20, 31]. However, only the paper [31] concerns the homogenization of thin linear shells. Unfortunately, we recognize that the results of this paper are incorrect due to errors made in the passage to a limit ($\varepsilon \rightarrow 0$). Also the local problems, assumed as being uncoupled, seem to be incorrect. For the Budiansky-Sanders model studied in [31] these problems should be coupled, cf. [40] and Sect. 3 below. We shall not follow the energy proof employed in this paper. Instead we use here asymptotic expansions. The problem studied is not conventional since not only do the components of the stiffness tensors depend on periodically varying elastic moduli but they also vary nonperiodically according to variations of metric tensors and curvature tensors of the shell mid-surface. Thus we are faced with a non-

uniform homogenization problem, cf. [8], pp. 71–87. The homogenized model of such a problem is still nonhomogeneous. The asymptotic method employed reveals that the constitutive equations of the homogenized shell model are coupled: tangent deformations contribute to moments while changes of curvature affect membrane forces. This phenomenon has been recently confirmed by the method of Γ -convergence [40]. It is worth noting here that the averaged constitutive equations for lattice shells found by Pshenichnov [36] are decoupled. In the light of our findings his results should thus be reexamined.

The second part of the paper deals with moderately large deflections of shallow shells with periodic structure within the framework of the theory by Mushtari–Marguerre, cf. [24, 35]. Also for this case we obtain the homogenization formulae using the asymptotic expansion method. In this shallow shell theory the tangent displacements do not affect changes of curvature, what implies that the homogenized constitutive equations are uncoupled.

The paper is illustrated by an example of a shallow shell periodic with respect to one curvilinear coordinate. In this case the basic cell problems can be analytically solved. Thus the explicit formulae for effective stiffnesses can be given.

2. Basic relations

A variety of mathematical models for linear and nonlinear shell behaviour have already been developed, see e.g. [6, 24, 25, 33, 35]. In the present contribution we study two practically important models of thin shells under the Kirchhoff–Love hypothesis. The first model is sometimes called Koiter’s linear model [9]. The second model will be the geometrically nonlinear shallow shell model [10, 35].

Before proceeding to the process of homogenization, we provide the indispensable information related to a description of such shells.

Let $\Omega \subset \mathbb{R}^2$ be a bounded sufficiently regular domain and $\Phi: \Omega \rightarrow S$ a mapping of class $C^3(\bar{\Omega})$, see [9, 15]. Here S denotes the middle surface of the undeformed shell and $\bar{\Omega}$ stands for the closure of Ω . The plane \mathbb{R}^2 containing Ω is referred to coordinates (ξ^α) , whereas \mathbb{R}^3 is referred to (x^i) ; $\alpha = 1, 2$; $i = 1, 2, 3$. Obviously we have $S = \Phi(\Omega) \subset \subset \mathbb{R}^3$. The vectors tangent to the coordinate lines are $\underline{a}_\alpha = \partial\Phi/\partial\xi^\alpha = \Phi_{,\alpha}$. The symmetric covariant metric tensor of the middle surface S is given as the scalar product

$$(2.1) \quad a_{\alpha\beta} = \Phi_{,\alpha} \cdot \Phi_{,\beta}.$$

If \mathbf{v} is the unit normal to S , then the covariant components $b_{\alpha\beta}$ of the curvature tensor $\mathbf{b} = (b_{\alpha\beta})$ are

$$(2.2) \quad b_{\alpha\beta} = \mathbf{v} \cdot \mathbf{a}_{\alpha,\beta} = -\mathbf{a}_\alpha \cdot \mathbf{v}_{,\beta}.$$

The contravariant metric tensor $(a^{\alpha\beta})$ satisfying the relation $a^{\alpha\beta}a_{\beta\gamma} = \delta_\gamma^\alpha$ is used to raise the indices. The Christoffel symbols of the undeformed shell middle surface S are given by

$$(2.3) \quad \Gamma_{\beta\gamma}^\alpha = a^{\alpha\lambda}\Gamma_{\lambda\beta\gamma}, \quad \Gamma_{\alpha\beta\gamma} = \frac{1}{2}(a_{\alpha\beta,\gamma} + a_{\alpha\gamma,\beta} - a_{\beta\gamma,\alpha}).$$

Throughout this paper the Lagrangean description of the deformation of the shell is consequently used.

By $\mathbf{U} = (\mathbf{u}, w) = (u_\alpha, w)$ we denote the displacement vector of the middle surface of the shell. Thus we may write

$$(2.4) \quad \mathbf{U} = u^\alpha \mathbf{a}_\alpha + w \mathbf{v}.$$

The linear Kirchhoff–Love shell model (in the version by Koiter) is described by the following strain-displacement relations:

$$(2.5) \quad \theta_{\alpha\beta}(\mathbf{u}, w) = \frac{1}{2} (u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} w,$$

$$(2.6) \quad \varrho_{\alpha\beta}(\mathbf{u}, w) = w_{|\alpha\beta} - c_{\alpha\beta} w + b_\alpha^\sigma u_{\sigma|\beta} + b_\beta^\sigma u_{\sigma|\alpha} + b_{\alpha|\beta}^\sigma u_\sigma.$$

The nonlinear shallow shell model by Mushtari and Marguerre obeys the following kinematic relationships:

$$(2.7) \quad \gamma_{\alpha\beta}(\mathbf{u}, w) = \theta_{\alpha\beta}(\mathbf{u}, w) + \frac{1}{2} w_{|\alpha} w_{|\beta},$$

$$(2.8) \quad \varkappa_{\alpha\beta}(w) = w_{|\alpha\beta}.$$

In the above relations the quantities not yet defined are

$$(2.9) \quad u_{\alpha|\beta} = u_{\alpha, \beta} - \Gamma_{\alpha\beta}^\lambda u_\lambda,$$

$$(2.10) \quad w_{|\alpha\beta} = w_{, \alpha\beta} - \Gamma_{\alpha\beta}^\lambda w_{, \lambda},$$

$$(2.11) \quad c_{\alpha\beta} = b_\alpha^\gamma b_{\gamma\beta}.$$

We note that only the strain tensor $\boldsymbol{\gamma}$ is nonlinear. If $\Phi = \text{identity}$, then Eqs. (2.7)–(2.8) reduce to the well-known von Kármán plate equations.

For both of the above models the stored energy function W is quadratic. Thus for Koiter's model we have

$$(2.12) \quad W(\boldsymbol{\theta}, \boldsymbol{\rho}) = \frac{1}{2} A^{\alpha\beta\lambda\mu} \theta_{\alpha\beta} \theta_{\lambda\mu} + \frac{1}{2} D^{\alpha\beta\lambda\mu} \varrho_{\alpha\beta} \varrho_{\lambda\mu},$$

where

$$(2.13) \quad A^{\alpha\beta\lambda\mu} = A^{\beta\alpha\lambda\mu} = A^{\lambda\mu\alpha\beta}, \quad D^{\alpha\beta\lambda\mu} = D^{\beta\alpha\lambda\mu} = D^{\lambda\mu\alpha\beta},$$

are membrane and bending stiffness tensors of a shell made from a material for which the surfaces $z = \text{const}$ are surfaces of material symmetry; the coordinate z is perpendicular to the shell middle surface. We observe that the fourth order tensors, depending on ξ , are not necessarily isotropic.

A relation similar to Eq. (2.12) holds in the nonlinear case.

The vector (f^α, f) stands for the loading distributed over the shell middle surface.

We make the following assumptions:

$$(2.14) \quad f^\alpha \in L^2(\Omega), \quad f \in L^2(\Omega), \quad A^{\alpha\beta\lambda\mu} \in L^\infty(\Omega), \quad D^{\alpha\beta\lambda\mu} \in L^\infty(\Omega),$$

$$(2.15) \quad A^{\alpha\beta\lambda\mu} t_{\alpha\beta} t_{\lambda\mu} \geq c_0 t_{\alpha\beta} t_{\lambda\mu}, \quad D^{\alpha\beta\lambda\mu} t_{\alpha\beta} t_{\lambda\mu} \geq c_1 t_{\alpha\beta} t_{\lambda\mu}, \quad \forall \mathbf{t} \in \mathbf{M}_s(\mathbf{R});$$

where c_0 and c_1 are positive constants while $\mathbf{M}_s(\mathbf{R})$ is the space of real symmetric 2×2 matrices.

For Koiter's shell model the constitutive relations are

$$(2.16) \quad N^{\alpha\beta} = \partial W / \partial \theta_{\alpha\beta} = A^{\alpha\beta\lambda\mu} \theta_{\lambda\mu},$$

$$(2.17) \quad M^{\alpha\beta} = \partial W / \partial \varrho_{\alpha\beta} = D^{\alpha\beta\lambda\mu} \varrho_{\lambda\mu}.$$

Here \mathbf{N} , \mathbf{M} denote the membrane force tensor and the bending moment tensor, respectively.

In the case of the shallow shell we have

$$(2.18) \quad N^{\alpha\beta} = \partial W / \partial \gamma_{\alpha\beta} = A^{\alpha\beta\lambda\mu} \gamma_{\lambda\mu},$$

$$(2.19) \quad M^{\alpha\beta} = \partial W / \partial \varkappa_{\alpha\beta} = D^{\alpha\beta\lambda\mu} \varkappa_{\lambda\mu}.$$

BERNADOU and CIARLET [9] solved the problem of existence and uniqueness of solutions for Koiter's shell. Similar problems for nonlinear shallow shells have been studied by BERNADOU and ODEN [10].

3. Homogenization of Koiter's version of the Kirchhoff-Love shell model

3.1. Formulation of the problem

The objective of this part of the paper is a static analysis of a shell of a structure periodic with respect to the fixed curvilinear coordinates $\xi = (\xi^\alpha)$, see Fig. 1. We assume that the reduced (according to the neglect of σ_{33} stresses) elastic moduli $C^{\alpha\beta\lambda\mu}(\xi, y_0)$ or the shell thickness $h(\xi, y_0)$ are slowly varying with respect to ξ and are $\varepsilon_0 Y$ -periodic with respect to the second variable $y_0 = \xi/\varepsilon_0$; here Y is a rectangle $(0, Y_1) \times (0, Y_2)$, while ε_0 is a positive number. We aim at constructing the homogenized model for the considered periodic shell under the assumption that the periodicity segments $\Phi(\varepsilon_0 Y)$ are of shapes of shallow shells. Modelling such a shell consists of two steps. Starting from the three-dimensional description, one should reduce the transverse dimension keeping ε_0 constant. Thus one arrives at the $\varepsilon_0 Y$ -periodic two-dimensional shell model. Then an ε -family of εY -periodic shells is to be considered. Upon homogenizing the equations of such shells, one arrives at the effective model for the initially considered $\varepsilon_0 Y$ -periodic shell. The first step of the model construction is realized by substituting in Eq. (2.12) the tensors \mathbf{A}_ε and \mathbf{D}_ε for the tensors \mathbf{A} and \mathbf{D} ; \mathbf{A}_ε and \mathbf{D}_ε being functions of two variables: ξ and $y = \xi/\varepsilon$, εY -periodic in y . Thus we write

$$(3.1) \quad A_\varepsilon^{\alpha\beta\lambda\mu} = A^{\alpha\beta\lambda\mu}(\xi, y), \quad D_\varepsilon^{\alpha\beta\lambda\mu} = D^{\alpha\beta\lambda\mu}(\xi, y).$$

The functions $A^{\alpha\beta\lambda\mu}(\xi, \cdot)$ and $D^{\alpha\beta\lambda\mu}(\xi, \cdot)$ are Y -periodic and are such that

$$A^{\alpha\beta\lambda\mu}(\xi, \cdot) \in L_{\text{loc}}^\infty(\mathbb{R}^2), \quad D^{\alpha\beta\lambda\mu}(\xi, \cdot) \in L_{\text{loc}}^\infty(\mathbb{R}^2).$$

Further on we shall consider the shell being transversely homogeneous, hence

$$(3.2) \quad \begin{aligned} A^{\alpha\beta\lambda\mu}(\xi, y) &= h(\xi, y) C^{\alpha\beta\lambda\mu}(\xi, y), \\ D^{\alpha\beta\lambda\mu}(\xi, y) &= \frac{1}{12} h^3(\xi, y) C^{\alpha\beta\lambda\mu}(\xi, y). \end{aligned}$$

The functions $C^{\alpha\beta\lambda\mu}(\cdot, y)$ describe ε -independent variations of the metric of the shell middle surface.

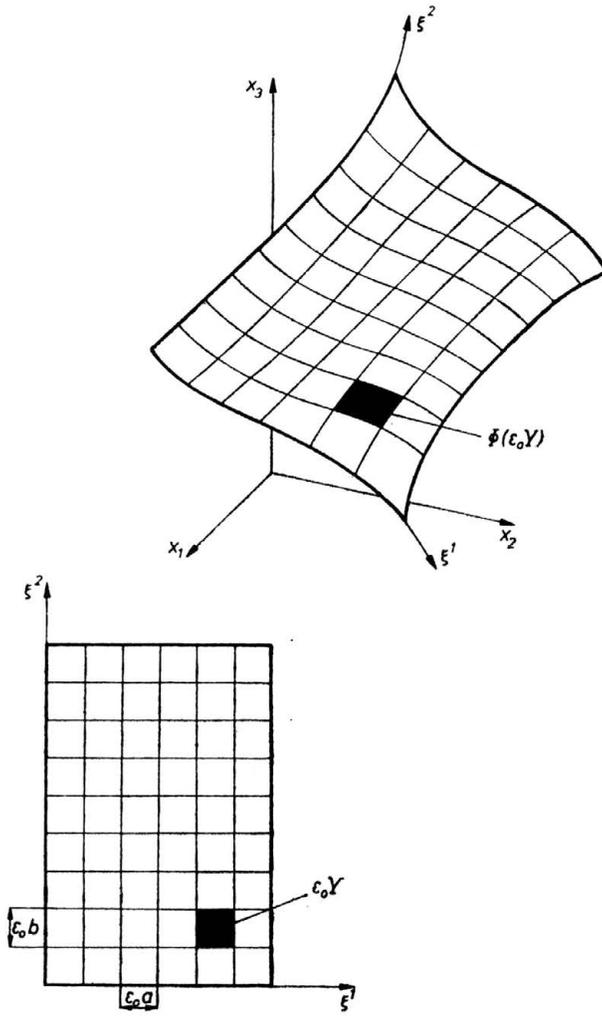


FIG. 1.

According to the relations (2.16) and (2.17) the constitutive equations for the ϵY -periodic shell have the form

$$(3.3) \quad N_\epsilon^{\alpha\beta} = A^{\alpha\beta\lambda\mu}(\xi, y)\theta_{\lambda\mu}^\epsilon, \quad M_\epsilon^{\alpha\beta} = D^{\alpha\beta\lambda\mu}(\xi, y)q_{\lambda\mu}^\epsilon,$$

where the deformation measures $\theta^\epsilon, \rho^\epsilon$ are associated with the unknown displacements $\mathbf{u}^\epsilon, w^\epsilon$ according to the relations (2.5) and (2.6). As it is usually done in papers on homogenization, we assume that the shell is clamped along its boundary. The shell is subject to external forces of densities (f^α, f) independent of the parameter ϵ .

The virtual work of stresses associated with the displacement field $(\mathbf{u}^\epsilon, w^\epsilon) \in V$ on deformations associated with virtual displacement fields $(\mathbf{v}^\epsilon, v^\epsilon) \in V$ reads

$$(3.4) \quad a^\epsilon(\mathbf{u}^\epsilon, w^\epsilon; \mathbf{v}^\epsilon, v^\epsilon) = \int_\Omega [N_\epsilon^{\alpha\beta}(v_{\alpha|\beta}^\epsilon - b_{\alpha\beta}v^\epsilon) + M_\epsilon^{\alpha\beta}(v_{|\alpha\beta}^\epsilon - c_{\alpha\beta}v^\epsilon + 2b_\alpha^\gamma v_{\gamma|\beta}^\epsilon + b_{\alpha|\beta}^\gamma v_\gamma^\epsilon)] \sqrt{a} d\xi,$$

where $a = \det[a_{\alpha\beta}]$ and $V = [H_0^1(\Omega)]^2 \times H_0^2(\Omega)$.

The virtual work of external forces has the form

$$(3.5) \quad f(\mathbf{v}^\varepsilon, v^\varepsilon) = \int_{\Omega} (f^\alpha v_\alpha^\varepsilon + f v^\varepsilon) \sqrt{ad\xi}.$$

Our main problem consists in finding $(\mathbf{u}^\varepsilon, w^\varepsilon) \in V$ such that

$$(3.6) \quad \alpha^\varepsilon(\mathbf{u}^\varepsilon, w^\varepsilon; \mathbf{v}^\varepsilon, v^\varepsilon) = f(\mathbf{v}^\varepsilon, v^\varepsilon), \quad \forall (\mathbf{v}^\varepsilon, v^\varepsilon) \in V.$$

The above problem will be referred to further as the (P^ε) problem. According to the results of BERNADOU and CIARLET [9], this problem is well-posed; $\mathbf{u}^\varepsilon, w^\varepsilon$ exist and are unique.

3.2. Derivation of the homogenization formulae

Similarly as in the case of nonuniform homogenization of the Dirichlet problem (cf. [8], pp. 71–87), we postulate that the solution $(\mathbf{u}^\varepsilon, w^\varepsilon)$ of the (P^ε) problem can be expanded in the form

$$(3.7) \quad \begin{aligned} u_\alpha^\varepsilon &= u_\alpha^{(0)}(\xi) + \varepsilon u_\alpha^{(1)}(\xi, y) + \dots, & y &= \xi/\varepsilon, \\ w^\varepsilon &= w^{(0)}(\xi) + \varepsilon^2 w^{(2)}(\xi, y) + \varepsilon^3 w^{(3)}(\xi, y) + \dots \end{aligned}$$

In similar way we expand also the trial functions $\mathbf{v}^\varepsilon, v^\varepsilon$. We assume that

$$(3.8) \quad \begin{aligned} (\mathbf{u}^{(0)}, w^{(0)}) &\in V; & u_\alpha^{(1)}, v_\alpha^{(1)} &\in C(\Omega \times Y), \\ u_\alpha^{(1)}(\xi, \cdot), v_\alpha^{(1)}(\xi, \cdot) &\in H_{\text{per}}^1(Y), \\ w^{(2)}, v^{(2)} &\in C(\Omega \times Y); & w^{(2)}(\xi, \cdot), v^{(2)}(\xi, \cdot) &\in H_{\text{per}}^2(Y), \end{aligned}$$

where, cf. [37]

$$H_{\text{per}}^1(Y) = \{v \in H^1(Y) \mid \text{traces of } v \text{ are equal at the opposite sides of } Y\},$$

$$H_{\text{per}}^2(Y) = \left\{ v \in H^2(Y) \mid \text{traces of } v \text{ and } \frac{\partial v}{\partial y_\alpha} \text{ are equal at the opposite sides of } Y \right\}.$$

We set

$$W_{\text{per}} = [H_{\text{per}}^1(Y)]^2 \times H_{\text{per}}^2(Y).$$

According to the relationships (3.3) and (2.5), (2.6), we can find stress and couple resultants associated with the displacement fields (3.7)

$$(3.9) \quad \begin{aligned} N_\varepsilon^{\alpha\beta} &= N_0^{\alpha\beta} + O(\varepsilon), & N_0^{\alpha\beta} &= A^{\alpha\beta\lambda\mu}(\xi, y) \left[\theta_{\lambda\mu}^h + \frac{\partial u_\lambda^{(1)}}{\partial y_\mu} \right], \\ M_\varepsilon^{\alpha\beta} &= M_0^{\alpha\beta} + O(\varepsilon), & M_0^{\alpha\beta} &= D^{\alpha\beta\lambda\mu}(\xi, y) \left[\varrho_{\lambda\mu}^h + \frac{\partial^2 w^{(2)}}{\partial y_\lambda \partial y_\mu} + 2b_\lambda^\alpha \frac{\partial u_\sigma^{(1)}}{\partial y_\mu} \right], \end{aligned}$$

where

$$\theta_{\alpha\beta}^h = \theta_{\alpha\beta}(\mathbf{u}^{(0)}, w^{(0)}), \quad \varrho_{\alpha\beta}^h = \varrho_{\alpha\beta}(\mathbf{u}^{(0)}, w^{(0)}).$$

Let \mathbf{N}_h and \mathbf{M}_h be averaged stress and couple resultants

$$(3.10) \quad \begin{aligned} N_h^{\alpha\beta}(\xi) &= \langle N_0^{\alpha\beta}(\xi, y) \rangle, \\ M_h^{\alpha\beta}(\xi) &= \langle M_0^{\alpha\beta}(\xi, y) \rangle. \end{aligned}$$

The parentheses $\langle \cdot \rangle$ imply averaging over Y :

$$(3.11) \quad \langle g \rangle = \frac{1}{|Y|} \int_Y g \, dy, \quad |Y| = \text{meas}(Y) = Y_1 \cdot Y_2.$$

Let us put $v_\alpha^\varepsilon = v_\alpha^{(0)}$, $v^\varepsilon = v^{(0)}$ into Eq. (3.6) and let ε tend to zero. According to the well-known lemma (cf. [8], p. 86), the integrands tend to their averages over Y . Thus the (P^ε) problem entails the homogenized problem (P_h) :

find $(\mathbf{u}^{(0)}, w^{(0)}) \in V$ such that $\forall (\mathbf{v}^{(0)}, v^{(0)}) \in V$

$$(3.12) \quad \int_\Omega [N_h^{\alpha\beta} (v_{\alpha|\beta}^{(0)} - b_{\alpha\beta} v^{(0)}) + M_h^{\alpha\beta} (v_{\alpha\beta}^{(0)} - c_{\alpha\beta} v^{(0)} + 2b_\alpha^\gamma v_{\gamma\beta}^{(0)} + b_{\alpha|\beta}^\gamma v_\gamma^{(0)})] \sqrt{a} \, d\xi = \int_\Omega (f^\alpha v_\alpha^{(0)} + f v^{(0)}) \sqrt{a} \, d\xi.$$

Now, let us put $v_\alpha^\varepsilon = v_\alpha^{(0)}(\xi) + \varepsilon v_\alpha^{(1)}(\xi, y) + \dots$ and $v^\varepsilon = v^{(0)}(\xi) + \varepsilon^2 v^{(2)}(\xi, y) + \dots$ into (3.6), then pass to zero with ε . Upon subtraction from the equation thus obtained of Eq. (3.12), one arrives at

$$(3.13) \quad \int_\Omega \left[\left\langle N_0^{\alpha\beta} \frac{\partial v_\alpha^{(1)}}{\partial y_\beta} \right\rangle + \left\langle M_0^{\alpha\beta} \left(\frac{\partial^2 v^{(2)}}{\partial y_\alpha \partial y_\beta} + 2b_\alpha^\gamma \frac{\partial v_\gamma^{(1)}}{\partial y_\beta} \right) \right\rangle \right] \sqrt{a} \, d\xi = 0,$$

$$\forall (\mathbf{v}^{(1)}(\xi, \cdot), v^{(2)}(\xi, \cdot)) \in W_{\text{per}}.$$

Let

$$\begin{aligned} v_\alpha^{(1)} &= v_\alpha(y) \varphi(\xi), & v_\alpha &\in H_{\text{per}}^1(Y), & \varphi &\in \mathcal{D}(\Omega), \\ v^{(1)} &= v(y) \psi(\xi), & v &\in H_{\text{per}}^2(Y), & \psi &\in \mathcal{D}(\Omega). \end{aligned}$$

Substituting into Eq. (3.13), we get

$$(3.14) \quad \int_\Omega \varphi \left\langle (N_0^{\gamma\beta} + 2b_\alpha^\gamma M_0^{\alpha\beta}) \frac{\partial v_\gamma}{\partial y_\beta} \right\rangle \sqrt{a} \, d\xi = 0, \quad \forall v_\alpha \in H_{\text{per}}^1(Y), \quad \forall \varphi \in \mathcal{D}(\Omega),$$

$$(3.15) \quad \int_\Omega \psi \left\langle M_0^{\alpha\beta} \frac{\partial^2 v}{\partial y_\alpha \partial y_\beta} \right\rangle \sqrt{a} \, d\xi = 0, \quad \forall v \in H_{\text{per}}^2(Y), \quad \psi \in \mathcal{D}(\Omega).$$

We eventually arrive at the local problem defined on the basic cell:

$$(P_{\text{loc}}) \left| \begin{aligned} &\text{find } (\mathbf{u}^{(1)}(\xi, \cdot), w^{(2)}(\xi, \cdot)) \in W_{\text{per}} \text{ such that} \\ &a_1(\mathbf{u}^{(1)}, w^{(2)}; \mathbf{v}, v) = \tilde{f}(\mathbf{v}, v) \quad \forall (\mathbf{v}, v) \in W_{\text{per}}, \end{aligned} \right.$$

provided that $\theta^h(\xi)$ and $\rho^h(\xi)$ are given.

We have introduced the following notations:

$$(3.16) \quad a_1(\mathbf{u}, w; \mathbf{v}, v) = \langle B^{\alpha\beta\lambda\mu} \varepsilon_{\alpha\beta}(\mathbf{u}) \varepsilon_{\lambda\mu}(\mathbf{v}) + E^{\alpha\beta\lambda\mu} (\varepsilon_{\alpha\beta}(\mathbf{u}) k_{\lambda\mu}(v) + \varepsilon_{\alpha\beta}(\mathbf{v}) k_{\lambda\mu}(w)) + D^{\alpha\beta\lambda\mu} k_{\alpha\beta}(w) k_{\lambda\mu}(v) \rangle,$$

$$(3.17) \quad \tilde{f}(\mathbf{v}, v) = - \left\langle A^{\alpha\beta\lambda\mu} \left(\delta_\alpha^\gamma \theta_{\lambda\mu}^h + \frac{h^2}{6} b_\alpha^\gamma \varrho_{\lambda\mu}^h \right) \varepsilon_{\gamma\beta}(\mathbf{v}) + D^{\alpha\beta\lambda\mu} \varrho_{\lambda\mu}^h k_{\alpha\beta}(v) \right\rangle,$$

where

$$(3.18) \quad \varepsilon_{\alpha\beta}(\mathbf{u}) = \frac{\partial u_\alpha}{\partial y_\beta}, \quad k_{\alpha\beta}(v) = \frac{\partial^2 v}{\partial y_\alpha \partial y_\beta}.$$

The tensors **B**, **E** and **D** are defined by

$$(3.19) \quad \begin{aligned} B^{\gamma\beta\alpha\mu}(\xi, y) &= \left[\delta_\alpha^\gamma \delta_\lambda^\sigma + \frac{h^2(\xi, y)}{3} b_\lambda^\gamma(\xi) b_\lambda^\sigma(\xi) \right] A^{\alpha\beta\lambda\mu}(\xi, y), \\ E^{\gamma\beta\lambda\mu}(\xi, y) &= \frac{(h^2(\xi, y))}{6} b_\alpha^\gamma(\xi) A^{\alpha\beta\lambda\mu}(\xi, y), \\ D^{\alpha\beta\lambda\mu}(\xi, y) &= \frac{1}{12} (h^2(\xi, y)) A^{\alpha\beta\lambda\mu}(\xi, y). \end{aligned}$$

The problem (P_{loc}) can be written in the form of the variational equalities:

find $(\mathbf{u}^{(1)}(\xi, \cdot), w^{(2)}(\xi, \cdot)) \in W_{per}$ such that

$$(3.20) \quad \begin{aligned} b(\mathbf{u}^{(1)}, \mathbf{v}) + e(\mathbf{v}, w^{(2)}) + f(\mathbf{v}) &= 0 \quad \forall \mathbf{v} \in [H_{per}^1(Y)]^2, \\ e(\mathbf{u}^{(1)}, v) + d(w^{(2)}, v) + g(v) &= 0 \quad \forall v \in H_{per}^2(Y), \end{aligned}$$

where

$$(3.21) \quad \begin{aligned} b(\mathbf{u}, \mathbf{v}) &= \left\langle B^{\gamma\beta\alpha\mu}(\xi, y) \frac{\partial u_\alpha}{\partial y_\mu} \frac{\partial v_\gamma}{\partial y_\beta} \right\rangle, \\ d(w, v) &= \left\langle D^{\alpha\beta\lambda\mu}(\xi, y) \frac{\partial^2 w}{\partial y_\lambda \partial y_\mu} \frac{\partial^2 v}{\partial y_\alpha \partial y_\beta} \right\rangle, \\ e(\mathbf{u}, w) &= \left\langle E^{\gamma\beta\lambda\mu}(\xi, y) \frac{\partial^2 w}{\partial y_\lambda \partial y_\mu} \frac{\partial u_\gamma}{\partial y_\beta} \right\rangle, \\ f(\mathbf{v}) &= \left\langle A^{\alpha\beta\lambda\mu}(\xi, y) \left(\delta_\alpha^\gamma \theta_{\lambda\mu}^h + \frac{h^2}{6} b_\alpha^\gamma \varrho_{\lambda\mu}^h \right) \frac{\partial v_\gamma}{\partial y_\beta} \right\rangle, \\ g(v) &= \left\langle D^{\alpha\beta\lambda\mu}(\xi, y) \varrho_{\lambda\mu}^h \frac{\partial^2 v}{\partial y_\alpha \partial y_\beta} \right\rangle. \end{aligned}$$

Note that the solutions $\mathbf{u}^{(1)}$, $w^{(2)}$ are linear with respect to θ^h and ρ^h , hence there exist functions $\Psi^{(\alpha\beta)}(\xi, y)$, $\Phi^{(\alpha\beta)}(\xi, y)$, $\Xi^{(\alpha\beta)}(\xi, y)$, $\chi^{(\alpha\beta)}(\xi, y)$, such that

$$(3.22) \quad \begin{aligned} \mathbf{u}^{(1)} &= \Psi^{(\alpha\beta)}(\xi, y) \theta_{\alpha\beta}^h(\xi) + \Phi^{(\alpha\beta)}(\xi, y) \varrho_{\alpha\beta}^h(\xi), \\ w^{(2)} &= \Xi^{(\alpha\beta)}(\xi, y) \theta_{\alpha\beta}^h(\xi) + \chi^{(\alpha\beta)}(\xi, y) \varrho_{\alpha\beta}^h(\xi). \end{aligned}$$

The functions $(\Psi^{(\alpha\beta)}(\xi, \cdot), \Xi^{(\alpha\beta)}(\xi, \cdot)) \in W_{per}$ are solutions to the problem

$$(P_{loc}^1) \quad \begin{cases} b(\Psi^{(\lambda\mu)}, \mathbf{v}) + e(\mathbf{v}, \Xi^{(\lambda\mu)}) + \left\langle A^{\alpha\beta\lambda\mu} \frac{\partial v_\alpha}{\partial y_\beta} \right\rangle = 0, \\ e(\Psi^{(\lambda\mu)}, v) + d(\Xi^{(\lambda\mu)}, v) = 0, \quad \forall (\mathbf{v}, v) \in W_{per}. \end{cases}$$

Similarly the functions $(\Phi^{(\lambda\mu)}(\xi, \cdot), \chi^{(\lambda\mu)}(\xi, \cdot)) \in W_{per}$ are solutions to the problem

$$(P_{loc}^2) \quad \begin{cases} b(\Phi^{(\lambda\mu)}, \mathbf{v}) + e(\mathbf{v}, \chi^{(\lambda\mu)}) + \left\langle \frac{h^2}{6} A^{\alpha\beta\lambda\mu} b_\alpha^\gamma \frac{\partial v_\gamma}{\partial y_\beta} \right\rangle = 0, \\ e(\Phi^{(\lambda\mu)}, v) + d(\chi^{(\lambda\mu)}, v) + \left\langle D^{\alpha\beta\lambda\mu} \frac{\partial^2 v}{\partial y_\alpha \partial y_\beta} \right\rangle = 0, \quad \forall (\mathbf{v}, v) \in W_{per}. \end{cases}$$

It is worth noting that the local problems (P_{loc}^1) and (P_{loc}^2) are coupled due to the influence of the geometry of the shell.

3.3. Well-posedness of the basic cell problem

Prior to analysing the homogenized problem, we shall prove that the problem (P_{loc}) is well-posed. Thus also the problems (P_{loc}^α) are well-posed.

Let

$$\tilde{W} = \{(\mathbf{v}, v) \in W_{\text{per}} | \langle \mathbf{v} \rangle = 0 \quad \text{and} \quad \langle v \rangle = 0\},$$

and

$$\mathcal{R} = \{(\mathbf{v}, v) \in \tilde{W} | a_1(\mathbf{v}, v; \mathbf{v}, v) = 0\}.$$

We shall demonstrate that the form $a_1(\cdot, \cdot; \cdot, \cdot)$ is \tilde{W} -elliptic and $\mathcal{R} = \{\mathbf{0}, 0\}$.

According to our assumption concerning the mapping Φ (see Sect. 2) there exists a constant $\lambda_0 > 0$ such that $|b_\alpha^\beta| < \lambda_0$. The bilinear form $a_1(\cdot, \cdot; \cdot, \cdot)$ can be written in the form

$$(3.23) \quad a_1(\mathbf{u}, w; \mathbf{u}, w) = \left\langle A^{\alpha\beta\lambda\mu} \left(\varepsilon_{\alpha\beta}(\mathbf{u}) \varepsilon_{\lambda\mu}(\mathbf{u}) + \frac{h^2}{12} K_{\alpha\beta} K_{\lambda\mu} \right) \right\rangle,$$

where

$$K_{\alpha\beta} = k_{\alpha\beta}(w) + b_\alpha^\gamma \varepsilon_{\gamma\beta}(\mathbf{u}) + b_\beta^\gamma \varepsilon_{\gamma\alpha}(\mathbf{u}) \quad \text{and} \quad (\mathbf{u}, w) \in W_{\text{per}}.$$

On account of the inequality (2.15)₁, we have

$$(3.24) \quad a_1(\mathbf{u}, w; \mathbf{u}, w) \geq c \left\langle \varepsilon_{(\alpha\beta)} \varepsilon_{(\alpha\beta)} + \frac{\bar{h}^2}{12} K_{\alpha\beta} K_{\alpha\beta} \right\rangle,$$

where c, \bar{h} are positive constants.

The condition $a_1(\mathbf{u}, w; \mathbf{u}, w) = 0$ implies $\varepsilon_{(\alpha\beta)} = 0$ and $K_{\alpha\beta} = 0$. The first equation yields

$$u_1 = a - by_2, \quad u_2 = d + by_1,$$

where a, b and d are constants.

Taking account of the Y -periodicity of the functions u_α and the condition $\langle u_\alpha \rangle = 0$, we obtain $u_\alpha = 0$. Hence $(\varepsilon_{\alpha\beta}(\mathbf{u})) = \nabla \mathbf{u} = 0$ and $k_{\alpha\beta}(w) = 0$. Thus $w = c_1 y_1 + c_2 y_2 + c_3$. Since $w \in H_{\text{per}}^2$ and $\langle w \rangle = 0$, therefore $w \equiv 0$. We see that $\mathcal{R} = \{\mathbf{0}, 0\}$.

Now we pass to proving the \tilde{W} -ellipticity of $a_1(\cdot, \cdot; \cdot, \cdot)$. We set $\|\cdot\| = \|\cdot\|_{L^2(\Gamma)}$. An elementary inequality furnishes (see [9], Chapter 6)

$$(3.25) \quad \begin{aligned} \|K_{11}\|^2 &\geq \frac{\beta}{1+\beta} \|k_{11}\|^2 - 4\lambda_0 \beta (\|\varepsilon_{11}\|^2 + \|\varepsilon_{21}\|^2), \\ \|K_{22}\|^2 &\geq \frac{\beta}{1+\beta} \|k_{22}\|^2 - 4\lambda_0 \beta (\|\varepsilon_{12}\|^2 + \|\varepsilon_{22}\|^2), \\ \|K_{12}\|^2 &\geq \frac{\beta}{1+\beta} \|k_{12}\|^2 - 8\lambda_0 \beta (\|\varepsilon_{11}\|^2 + \|\varepsilon_{22}\|^2 + \|\varepsilon_{12}\|^2 + \|\varepsilon_{21}\|^2), \end{aligned}$$

where $\beta > 0$.

Taking account of the inequalities (3.25) in the inequality (3.24), we arrive at

$$(3.26) \quad a_1(\mathbf{u}, w; \mathbf{u}, w) \geq c_1 (\varepsilon(\mathbf{u}) + k(w) - c_2 \beta \|\nabla \mathbf{u}\|^2),$$

where

$$c_1 > 0, c_2 > 0, \quad \varepsilon(\mathbf{u}) = \|\varepsilon_{(\alpha\beta)}(\mathbf{u})\| \quad \text{and} \quad k(w) = \|k_{\alpha\beta}(w)k_{\alpha\beta}(w)\|.$$

Bearing in mind that $\mathcal{R} = \{0, 0\}$, we conclude that $\sqrt{\varepsilon(\mathbf{u})}$ and $\sqrt{k(w)}$ are equivalent to $\|\mathbf{u}\|_{H^1(\mathcal{Y})}$, $\|w\|_{H^2(\mathcal{Y})}$, respectively, provided that $(\mathbf{u}, w) \in \tilde{W}$, see [16, 17]. Further we have $\|\nabla \mathbf{u}\|^2 \leq \|\mathbf{u}\|_{H^1(\mathcal{Y})}^2$.

For $\beta > 0$ sufficiently small, we finally obtain

$$(3.27) \quad a_1(\mathbf{u}, w; \mathbf{u}, w) \geq c(\|\mathbf{u}\|_{H^1(\mathcal{Y})} + \|w\|_{H^2(\mathcal{Y})}),$$

where

$$(\mathbf{u}, w) \in \tilde{W}.$$

On the other hand, the functions $A^{\alpha\beta\lambda\mu}$ and $D^{\alpha\beta\lambda\mu}$ are essentially bounded. Thus the linear form $\tilde{f}(\mathbf{u}, w)$ is continuous. By employing the Lax–Milgram theorem, we infer that the local problem (P_{loc}) possesses a unique solution in \tilde{W} .

3.4. Properties of the homogenized problem

Now we pass to the study of the homogenized constitutive equations. Substitution of Eqs. (3.22) into Eqs. (3.10) leads to

$$(3.28) \quad N_h^{\alpha\beta} = A_h^{\alpha\beta\lambda\mu} \theta_{\lambda\mu}^h + E_h^{\alpha\beta\lambda\mu} \varrho_{\lambda\mu}^h,$$

$$(3.29) \quad M_h^{\alpha\beta} = F_h^{\alpha\beta\lambda\mu} \theta_{\lambda\mu}^h + D_h^{\alpha\beta\lambda\mu} \varrho_{\lambda\mu}^h,$$

where

$$(3.30) \quad \begin{aligned} A_h^{\alpha\beta\delta\gamma} &= \left\langle A^{\alpha\beta\lambda\mu} \left(\delta_\lambda^\gamma \delta_\mu^\delta + \frac{\partial \Psi_\lambda^{(\delta\gamma)}}{\partial y_\mu} \right) \right\rangle, \\ E_h^{\alpha\beta\delta\gamma} &= \left\langle A^{\alpha\beta\lambda\mu} \frac{\partial \Phi_\lambda^{(\delta\gamma)}}{\partial y_\mu} \right\rangle, \\ F_h^{\alpha\beta\delta\gamma} &= \left\langle \frac{h^2}{12} A^{\alpha\beta\lambda\mu} \left(2b_\lambda^\sigma \frac{\partial \Psi_\sigma^{(\delta\gamma)}}{\partial y_\mu} + \frac{\partial^2 \Xi^{(\delta\gamma)}}{\partial y_\lambda \partial y_\mu} \right) \right\rangle, \\ D_h^{\alpha\beta\delta\gamma} &= \left\langle \frac{h^2}{12} A^{\alpha\beta\lambda\mu} \left(\delta_\lambda^\delta \delta_\mu^\gamma + 2b_\lambda^\sigma \frac{\partial \Phi_\sigma^{(\delta\gamma)}}{\partial y_\mu} + \frac{\partial^2 \chi^{(\delta\gamma)}}{\partial y_\lambda \partial y_\mu} \right) \right\rangle. \end{aligned}$$

The effective tensors satisfy the following symmetry conditions:

$$(3.31) \quad A_h^{\alpha\beta\delta\gamma} = A_h^{\delta\gamma\alpha\beta}, \quad D_h^{\alpha\beta\delta\gamma} = D_h^{\delta\gamma\alpha\beta},$$

$$(3.32) \quad F_h^{\delta\gamma\alpha\beta} = E_h^{\alpha\beta\delta\gamma}.$$

To prove it, let us take $\mathbf{v} = \Phi^{(\delta\gamma)}$ and $v = \chi^{(\delta\gamma)}$ in (P_{loc}^2) and $\mathbf{v} = \Psi^{(\lambda\mu)}$, $v = \Xi^{(\lambda\mu)}$ in (P_{loc}^1) . Then we obtain

$$(3.33) \quad b(\Psi^{(\lambda\mu)}, \Phi^{(\delta\gamma)}) + e(\Phi^{(\delta\gamma)}, \Xi^{(\lambda\mu)}) + \left\langle A^{\alpha\beta\lambda\mu} \frac{\partial \Phi_\alpha^{(\delta\gamma)}}{\partial y_\beta} \right\rangle = 0,$$

$$(3.34) \quad e(\Psi^{(\lambda\mu)}, \chi^{(\delta\gamma)}) = -d(\Xi^{(\lambda\mu)}, \chi^{(\delta\gamma)}),$$

$$(3.35) \quad b(\Phi^{(\delta\gamma)}, \Psi^{(\lambda\mu)}) + e(\Psi^{(\lambda\mu)}, \chi^{(\delta\gamma)}) + \left\langle \frac{h^2}{6} A^{\alpha\beta\delta\gamma} b_\alpha^\sigma \frac{\partial \Psi_\sigma^{(\lambda\mu)}}{\partial y_\beta} \right\rangle = 0,$$

$$(3.36) \quad e(\Phi^{(\delta\gamma)}, \Xi^{(\lambda\mu)}) + d(\chi^{(\delta\gamma)}, \Xi^{(\lambda\mu)}) + \left\langle D^{\alpha\beta\delta\gamma} \frac{\partial^2 \Xi^{(\lambda\mu)}}{\partial y_\alpha \partial y_\beta} \right\rangle = 0.$$

Equation (3.33) implies

$$(3.37) \quad E_h^{\lambda\mu\delta\gamma} = -b(\Psi^{(\lambda\mu)}, \Phi^{(\delta\gamma)}) - e(\Phi^{(\delta\gamma)}, \Xi^{(\lambda\mu)}).$$

On adding Eqs. (3.35) and (3.36) and taking into account Eq. (3.30)₃, one obtains

$$(3.38) \quad F_h^{\delta\gamma\lambda\mu} + b(\Phi^{(\delta\gamma)}, \Psi^{(\lambda\mu)}) + e(\Psi^{(\lambda\mu)}, \chi^{(\delta\gamma)}) + e(\Phi^{(\delta\gamma)}, \Xi^{(\lambda\mu)}) + d(\chi^{(\delta\gamma)}, \Xi^{(\lambda\mu)}) = 0.$$

Substituting Eq. (3.34) into Eq. (3.38), we arrive at

$$(3.39) \quad F^{\delta\gamma\lambda\mu} = -b(\Phi^{(\delta\gamma)}, \Psi^{(\lambda\mu)}) - e(\Phi^{(\delta\gamma)}, \Xi^{(\lambda\mu)}).$$

Now we see that Eqs. (3.37) and (3.39) imply the symmetry property (3.32) since the form $b(\cdot, \cdot)$ is symmetric.

In order to prove the conditions (3.31), let us define new symmetric matrices:

$$(3.40) \quad \hat{A}_h^{\alpha\beta\delta\gamma} = \left\langle A^{ve\lambda\mu} \left(\delta_\lambda^\gamma \delta_\mu^\delta + \frac{\partial \Psi_\lambda^{(\delta\gamma)}}{\partial y_\mu} \right) \left(\delta_\nu^\alpha \delta_\rho^\beta + \frac{\partial \Psi_\nu^{(\alpha\beta)}}{\partial y_\rho} \right) \right\rangle,$$

$$\hat{D}_h^{\alpha\beta\delta\gamma} = \left\langle D^{ve\lambda\mu} \left(\delta_\lambda^\gamma \delta_\mu^\delta + \frac{\partial^2 \chi^{(\delta\gamma)}}{\partial y_\lambda \partial y_\mu} \right) \left(\delta_\nu^\alpha \delta_\rho^\beta + \frac{\partial^2 \chi^{(\alpha\beta)}}{\partial y_\nu \partial y_\rho} \right) \right\rangle.$$

Let us take $\mathbf{v} = \Psi^{(\alpha\beta)}$ in the first equation of (P_{1oc}^1) and $v = \Xi^{(\alpha\beta)}$ in the second equation of (P_{1oc}^1) . On combining these equations, one obtains

$$(3.41) \quad A_h^{\alpha\beta\delta\gamma} = \hat{A}^{\alpha\beta\delta\gamma} - d(\Xi^{(\alpha\beta)}, \Xi^{(\delta\gamma)}) + g(\Psi^{(\alpha\beta)}, \Psi^{(\delta\gamma)}),$$

where

$$g(\mathbf{u}, \mathbf{v}) = \left\langle \frac{h^2}{3} b_\alpha^\gamma b_\lambda^\sigma A^{\alpha\beta\lambda\mu} \frac{\partial u_\sigma}{\partial y_\mu} \frac{\partial v_\gamma}{\partial y_\beta} \right\rangle,$$

which proves the relation (3.31)₁ since the forms $g(\cdot, \cdot)$ and $d(\cdot, \cdot)$ are symmetric.

Similarly it can readily be shown that

$$(3.42) \quad D_h^{\alpha\beta\delta\gamma} = \hat{D}_h^{\alpha\beta\delta\gamma} - b(\Phi^{(\alpha\beta)}, \Phi^{(\delta\gamma)}),$$

which proves the condition (3.31)₂ due to the symmetry of the bilinear form $b(\cdot, \cdot)$.

The elastic potential of the homogenized shell is given by

$$(3.43) \quad W^h(\boldsymbol{\theta}^h, \boldsymbol{\rho}^h) = \frac{1}{2} (N_h^{\alpha\beta} \theta_{\alpha\beta}^h + M_h^{\alpha\beta} \varrho_{\alpha\beta}^h),$$

or

$$(3.44) \quad W^h(\boldsymbol{\theta}^h, \boldsymbol{\rho}^h) = \frac{1}{2} \langle A^{\alpha\beta\lambda\mu} \bar{\theta}_{\alpha\beta} \bar{\theta}_{\lambda\mu} + D^{\alpha\beta\lambda\mu} \bar{\varrho}_{\alpha\beta} \bar{\varrho}_{\lambda\mu} \rangle,$$

where

$$(3.45) \quad \bar{\theta}_{\lambda\mu} = \theta_{\lambda\mu}^h + \frac{\partial \Psi_\lambda^{(\delta\gamma)}}{\partial y_\mu} \theta_{\delta\gamma}^h + \frac{\partial \Phi_\lambda^{(\delta\gamma)}}{\partial y_\mu} \varrho_{\delta\gamma}^h,$$

$$(3.46) \quad \bar{\varrho}_{\lambda\mu} = \varrho_{\lambda\mu}^h + \frac{\partial^2 \Xi^{(\delta\gamma)}}{\partial y_\lambda \partial y_\mu} \theta_{\delta\gamma}^h + \frac{\partial^2 \chi^{(\delta\gamma)}}{\partial y_\lambda \partial y_\mu} \varrho_{\delta\gamma}^h + 2b_\lambda^\sigma \frac{\partial \Psi_\sigma^{(\delta\gamma)}}{\partial y_\mu} \theta_{\delta\gamma}^h + 2b_\lambda^\sigma \frac{\partial \Phi_\sigma^{(\delta\gamma)}}{\partial y_\mu} \varrho_{\delta\gamma}^h,$$

and

$$\boldsymbol{\theta}^h \in M_s(\mathbb{R}), \quad \boldsymbol{\rho}^h \in M_s(\mathbb{R}).$$

The formula (3.44) can be verified by employing the equations describing the local problems (P_{loc}^α).

By using Eq. (3.44) and the inequalities (2.15) we readily get

$$(3.47) \quad W^h \geq c \left\langle \bar{\theta}_{\alpha\beta} \bar{\theta}_{\alpha\beta} + \frac{h^2}{12} \bar{\varrho}_{\alpha\beta} \bar{\varrho}_{\alpha\beta} \right\rangle,$$

where $c > 0$ is a constant. Hence we infer that W^h is non-negative. Now we shall prove that $W^h = 0$ implies $\theta^h = 0$ and $\rho^h = 0$. Let $W^h = 0$. Then $\bar{\theta}^h = 0$ and $\bar{\rho} = 0$, hence $\langle \bar{\theta} \rangle = 0$, and $\langle \bar{\rho} \rangle = 0$. Bearing in mind the periodicity properties of the functions $\Psi^{(\alpha\beta)}(\xi, \cdot)$, $\Phi^{(\alpha\beta)}(\xi, \cdot)$, $\Xi^{(\alpha\beta)}(\xi, \cdot)$ and $\chi^{(\alpha\beta)}(\xi, \cdot)$ we conclude that $\langle \theta_{\alpha\beta}^h \rangle = \theta_{\alpha\beta}^h = 0$ and similarly $\varrho_{\alpha\beta}^h = 0$. Thus we eventually infer that there exists a constant $C > 0$ such that

$$(3.48) \quad W^h(\theta^h, \rho^h) \geq C(\theta_{\alpha\beta}^h \theta_{\alpha\beta}^h + \varrho_{\alpha\beta}^h \varrho_{\alpha\beta}^h),$$

for each

$$\theta^h \in M_s(\mathbb{R}), \quad \rho^h \in M_s(\mathbb{R}).$$

By applying the results due to BERNADOU and CIARLET [9], we infer that a solution $(u^{(0)}, w^{(0)})$ of the homogenized problem exists and is unique.

REMARK 3.1

The presented method of constructing the effective model for the $\varepsilon_0 Y$ -periodic shell described in Sect. 3.1 is a generalization of the method by CAILLERIE [11, 12] and KOHN and VOGELIUS [21, 22] to the case of shells. Note here that the model obtained applies neither to shells whose periodicity segments $\Phi(\varepsilon_0 Y)$ are of shapes of curvilinear prisms nor to the case when these segments are slender. To describe the former shells, one should apply the simultaneous passage to a limit ($h \rightarrow 0, \varepsilon \rightarrow 0$). For the latter, one should carry out the process of a reduction of the transverse dimension after homogenization.

4. Homogenization of a nonlinear shallow shell model

Similarly as in Sect. 3 we are to analyse a shell of εY -periodic stiffnesses $A_\varepsilon, D_\varepsilon$ with respect to curvilinear parametrization (ξ^α) . The behaviour of the shell will be analysed within the framework of the Mushtari–Marguerre shell model. The constitutive relations are assumed here in accordance with the relations (2.18) and (2.19)

$$(4.1) \quad N_\varepsilon^{\alpha\beta} = A^{\alpha\beta\lambda\mu}(\xi, y) \gamma_{\lambda\mu}^\varepsilon, \quad M_\varepsilon^{\alpha\beta} = D^{\alpha\beta\lambda\mu}(\xi, y) \kappa_{\lambda\mu}^\varepsilon, \quad y = \xi/\varepsilon.$$

The deformation measures γ^ε and κ^ε are associated with the displacement fields $u^\varepsilon, w^\varepsilon$ according to the relationships (2.7), (2.8). As in Sect. 3, we assume that the shell is clamped along its boundary. The loads (f^α, f) are assumed to be ε -independent.

The variational formulation of the considered boundary value problem reads:

$$(4.2) \quad (P_N^\varepsilon) \quad \left| \begin{array}{l} \text{find } (u^\varepsilon, w^\varepsilon) \in V \text{ such that} \\ a_N^\varepsilon(u^\varepsilon, w^\varepsilon; v^\varepsilon, v^\varepsilon) = f(v^\varepsilon, v^\varepsilon) \end{array} \right. \quad \forall (v^\varepsilon, v^\varepsilon) \in V,$$

where

$$(4.3) \quad a_N^\varepsilon(u, w; v, v) = \int_\Omega [N_\varepsilon^{\alpha\beta} \eta_{\alpha\beta}(w, v, v) + M_\varepsilon^{\alpha\beta} \kappa_{\alpha\beta}(v)] \sqrt{ad} d\xi,$$

$$(4.4) \quad \eta_{\alpha\beta}(w, \mathbf{v}, v) = v_{|\alpha|\beta} - b_{\alpha\beta}v + \frac{1}{2}(w_{|\alpha}v_{|\beta} + w_{|\beta}v_{|\alpha}).$$

The linear form $f(\cdot, \cdot)$ is defined by Eq. (3.5) while $\kappa_{\alpha\beta}(\cdot)$ by the relationship (2.8).

Sufficient conditions for the existence and independent conditions for the uniqueness of solutions of the (P_N^e) problem have been put forward in the paper [10] by BERNADOU and ODEN.

The solution (\mathbf{u}^e, w^e) to the (P_N^e) problem is sought in the form (3.7). Similarly, we expand the trial functions (\mathbf{v}^e, v^e) . Then we infer that the nonlinear terms in the strain-displacement relations do not influence the asymptotic process. This process runs similarly as it has been described in the previous section. Thus we shall not go here into details in order to avoid repetitions. We shall report below only the main results of the asymptotic homogenization procedure.

The homogenized problem reads:

$$(4.5) \quad (P_N^h) \quad \left\{ \begin{array}{l} \text{find } (\mathbf{u}^{(0)}, w^{(0)}) \in V \quad \text{such that} \\ a_N^h(\mathbf{u}^0, w^0, \mathbf{v}, v) = f(\mathbf{v}, v) \end{array} \right. \quad \forall (\mathbf{v}, v) \in V,$$

where

$$(4.6) \quad a_N^h(\mathbf{u}, w; \mathbf{v}, v) = \int_{\Omega} [N_h^{\alpha\beta} \eta_{\alpha\beta}(w, \mathbf{v}, v) + M_h^{\alpha\beta} \kappa_{\alpha\beta}(v)] \sqrt{ad} d\xi,$$

$$(4.7) \quad N_h^{\alpha\beta} = A_h^{\alpha\beta\lambda\mu}(\xi) \gamma_{\lambda\mu}^h, \quad M_h^{\alpha\beta} = D_h^{\alpha\beta\lambda\mu}(\xi) \chi_{\lambda\mu}^h.$$

Here the deformation measures γ^h, χ^h are associated with the displacement fields $(\mathbf{u}^{(0)}, w^{(0)})$ according to the relations (2.7) and (2.8). The effective tensors $\mathbf{A}_h, \mathbf{D}_h$ are determined by the auxiliary functions $\Psi^{(\gamma^\delta)}(\xi, y), \chi^{(\gamma^\delta)}(\xi, y)$. The latter are now solutions of the following *independent basic cell problems*:

$$(4.8) \quad (P_{loc}^1) \quad \left\{ \begin{array}{l} \text{find } \Psi^{(\gamma^\delta)}(\xi, \cdot) \in [H_{\text{per}}^1(Y)]^2 \quad \text{such that} \\ \left\langle A^{\alpha\beta\sigma\mu}(\xi, y) \left[\delta_\sigma^\gamma \delta_\mu^\delta + \frac{\partial \Psi_\sigma^{(\gamma^\delta)}}{\partial y_\mu} \right] \frac{\partial v_\alpha}{\partial y_\beta} \right\rangle = 0, \quad \forall \mathbf{v} \in [H_{\text{per}}^1(Y)]^2, \end{array} \right.$$

$$(4.9) \quad (P_{loc}^2) \quad \left\{ \begin{array}{l} \text{find } \chi^{(\gamma^\delta)}(\xi, \cdot) \in H_{\text{per}}^2(Y) \quad \text{such that} \\ \left\langle D^{\alpha\beta\lambda\mu}(\xi, y) \left[\delta_\lambda^\gamma \delta_\mu^\delta + \frac{\partial^2 \chi^{(\gamma^\delta)}}{\partial y_\lambda \partial y_\mu} \right] \frac{\partial^2 v}{\partial y_\alpha \partial y_\beta} \right\rangle = 0, \quad \forall v \in H_{\text{per}}^2(Y). \end{array} \right.$$

The above local problems are similar to the basic cell problems occurring in the theory of homogenization of von Kármán plates, cf. [17]. Hence we infer that the problems (P_{loc}^α) possess solutions determined up to an additive constant, now depending additionally on ξ .

The effective stiffnesses are given by

$$(4.10) \quad A_h^{\alpha\beta\gamma\lambda}(\xi) = \left\langle A^{\alpha\beta\sigma\mu}(\xi, y) \left[\delta_\sigma^\gamma \delta_\mu^\lambda + \frac{\partial \Psi_\sigma^{(\gamma^\lambda)}}{\partial y_\mu} \right] \right\rangle,$$

$$(4.11) \quad D_h^{\alpha\beta\gamma\lambda}(\xi) = \left\langle D^{\alpha\beta\sigma\mu}(\xi, y) \left[\delta_\sigma^\gamma \delta_\mu^\lambda + \frac{\partial^2 \chi^{(\gamma^\lambda)}}{\partial y_\sigma \partial y_\mu} \right] \right\rangle.$$

It is easy to ascertain that the above formulae can be rearranged to the form (3.40), hence the symmetry conditions (3.31) are preserved. Moreover, one can prove that the homogenized potential

$$(4.12) \quad W^h = \frac{1}{2} (N_h^{\alpha\beta} \gamma_{\alpha\beta}^h + M_h^{\alpha\beta} \varkappa_{\alpha\beta}^h)$$

possesses the property (3.48).

Let us emphasize that the homogenized relations (4.7) are decoupled, similarly as the original relations (4.1). Homogenization does not change here the form of the constitutive relations. We conclude that the geometric von Kármán-type nonlinearity does not entail the coupling of the constitutive relations.

The functions $\Psi^{(\gamma\delta)}$ and $\chi^{(\gamma\delta)}$ make it possible to determine higher order terms of the asymptotic expansions, namely

$$(4.13) \quad \begin{aligned} u_a^{(1)} &= \Psi_a^{(\gamma\lambda)}(\xi, \xi/\varepsilon) \gamma_{\gamma\lambda}^h, \\ w^{(2)} &= \chi^{(\gamma\lambda)}(\xi, \xi/\varepsilon) \varkappa_{\gamma\lambda}^h. \end{aligned}$$

The above formulae are in a certain sense particular cases of Eq. (3.22).

5. Example: homogenized stiffnesses for the Mushtari–Marguerre shell with elastic characteristics periodic in one coordinate

Consider a Mushtari–Marguerre shell with stiffnesses $A_\varepsilon^{\alpha\beta\lambda\mu}(\xi, \xi/\varepsilon)$, $D_\varepsilon^{\alpha\beta\lambda\mu}(\xi, \xi/\varepsilon)$ being εa -periodic in the ξ^1 direction with respect to the second variable. Thus

$$\mathbf{A}_\varepsilon = \mathbf{A}(\xi, y_1), \quad \mathbf{D}_\varepsilon = \mathbf{D}(\xi, y_1), \quad y_1 = \xi^1/\varepsilon,$$

where $\mathbf{A}(\xi, \cdot)$ and $\mathbf{D}(\xi, \cdot)$ are a -periodic.

We can find now the exact solutions to the local problems (P_{loc}^α) formulated in Sect. 4. This is a rather simple exercise of homogenization, thus only the final results will be reported. Let us define the auxiliary functions of the argument ξ (to shorten notations this dependence is suppressed in the expressions below)

$$\begin{aligned} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} &= \int_0^a \begin{bmatrix} A^{1212} \\ -A^{1112} \\ A^{1111} \end{bmatrix} \frac{dy_1}{A}, \quad A = A^{1111} A^{1212} - (A^{1112})^2, \\ \delta &= \int_0^a [A^{1212} A^{1122} - A^{1112} A^{2212}] \frac{dy_1}{A}, \\ \varrho &= \int_0^a [A^{1111} A^{2212} - A^{1112} A^{1122}] \frac{dy_1}{A}, \\ d &= \alpha\gamma - \beta^2, \quad e = \gamma\delta^2 + \alpha\varrho^2 - 2\beta\delta\varrho, \\ C &= 2A^{1112} A^{1122} A^{2212} - A^{1212} (A^{1122})^2 - A^{1111} (A^{2212})^2. \end{aligned}$$

The effective membrane stiffnesses are

$$\begin{aligned} A_h^{1111} &= \frac{a\gamma}{d}, & A_h^{1112} &= A_h^{1211} = \frac{-a\beta}{d}, \\ A_h^{1122} &= A_h^{2211} = \frac{\gamma\delta - \beta\varrho}{d}, & A_h^{2212} &= A_h^{1222} = \frac{\alpha\varrho - \beta\delta}{d}, \\ A_h^{1212} &= A_h^{2121} = \frac{a\alpha}{d}, \\ A_h^{2222} &= \frac{1}{a} \int_0^a \left[A^{2222} + \frac{C}{A} \right] dy_1 + \frac{e}{ad}. \end{aligned}$$

Obviously the stiffnesses $A_h^{\alpha\beta\lambda\mu}$ depend on ξ but this dependence has been suppressed. Prior to reporting the effective bending stiffnesses, let us define auxiliary quantities:

$$\begin{aligned} D_1 &= \int_0^a \frac{dy_1}{D^{1111}}, & D_2 &= \int_0^a \frac{D^{1122}}{D^{1111}} dy_1, \\ D_3 &= \int_0^a \frac{D^{1112}}{D^{1111}} dy_1, & D_4 &= \frac{1}{a} \int_0^a \left(D^{2212} - \frac{D^{1112}D^{2212}}{D^{1111}} \right) dy_1, \\ D_5 &= \frac{1}{a} \int_0^a \left[D^{1212} - \frac{(D^{1112})^2}{D^{1111}} \right] dy_1, \\ D_6 &= \frac{1}{a} \int_0^a \left[D^{2222} - \frac{(D^{1122})^2}{D^{1111}} \right] dy_1. \end{aligned}$$

The effective bending stiffnesses are

$$\begin{aligned} D_h^{1111} &= \frac{a}{D_1}, & D_h^{1112} &= D_h^{1211} = \frac{D_3}{D_1}, & D_h^{1122} &= D_h^{2211} = \frac{D_2}{D_1}, \\ D_h^{2212} &= D_4 + \frac{D_2 D_3}{aD_1}, & D_h^{1212} &= D_5 + \frac{(D_3)^2}{aD_1}, \\ D_h^{2222} &= D_6 + \frac{(D_2)^2}{aD_1}. \end{aligned}$$

We recall that the above effective stiffnesses depend on ξ since the homogenized model is nonhomogeneous. The above formulae are shorter in the orthotropic case for which $A^{1112} = A^{1222} = 0$ and $D^{1112} = D^{2221} = 0$.

Because of its practical importance, it is worth displaying the corresponding results explicitly.

The membrane effective stiffnesses are now given by

$$A_h^{1111} = \left[\frac{1}{a} \int_0^a \frac{dy_1}{A^{1111}} \right]^{-1}, \quad A_h^{1122} = \left(\int_0^a \frac{A^{1122}}{A^{1111}} dy_1 \right) \left(\int_0^a \frac{dy_1}{A^{1111}} \right)^{-1},$$

$$A_h^{1212} = \left[\frac{1}{a} \int_0^a \frac{dy_1}{A^{1212}} \right]^{-1},$$

$$A_h^{2222} = \frac{1}{a} \int_0^a \left[A^{2222} - \frac{(A^{1122})^2}{A^{1111}} \right] dy_1 + \frac{1}{a} \left(\int_0^a \frac{A^{1122}}{A^{1111}} dy_1 \right)^2 \left(\int_0^a \frac{dy_1}{A^{1111}} \right)^{-1},$$

$$A_h^{1222} = A_h^{2111} = 0.$$

The bending effective stiffnesses are

$$D_h^{1111} = \left[\frac{1}{a} \int_0^a \frac{dy_1}{D^{1111}} \right]^{-1}, \quad D_h^{1122} = \left(\int_0^a \frac{D^{1122}}{D^{1111}} dy_1 \right) \left(\int_0^a \frac{dy_1}{D^{1111}} \right)^{-1},$$

$$D_h^{1212} = \frac{1}{a} \int_0^a D^{1212} dy_1,$$

$$D_h^{2222} = \frac{1}{a} \int_0^a \left[D^{2222} - \frac{(D^{1122})^2}{D^{1111}} \right] dy_1 + \frac{1}{a} \left(\int_0^a \frac{D^{1122}}{D^{1111}} dy_1 \right)^2 \left(\int_0^a \frac{dy_1}{D^{1111}} \right)^{-1},$$

$$D_h^{1222} = D_h^{2111} = 0.$$

The difference between the formulae for A_h^{1212} and D_h^{1212} (the other formulae are of identical form) follows from the fact that the flexural behaviour of the shell is constrained by the Kirchhoff-Love assumptions. If we start from a shallow shell model with transverse shear deformations allowed, then the effective stiffness D_h^{1212} will be defined by the formula

$$\bar{D}_h^{1212} = \left(\frac{1}{a} \int_0^a \frac{dy_1}{D^{1212}} \right)^{-1}.$$

This fact has been recognized in the paper [29] concerning Reissner-like plates.

It is worth noting that the formulae derived in this section are applicable only when the cell of periodicity is a shallow shell itself.

6. Concluding remarks

The asymptotic method applied in this paper was efficient by virtue of the fact that we have worked with variational formulations. Working with strong formulations would be a formidable task because of a great complexity of shell equations.

Let us call attention to the fact that our results are very "sensitive" to the definition of the changes of the curvature tensor. As it is known from the ample literature on the theory of shells, there exists a great number of alternative versions of the Kirchhoff-Love theory which differ in the definition of this deformation measure, cf. [6, 23], see also [25]. For instance, in the model by KOITER (1960, cf. [23]) we have

$$Q_{\alpha\beta} = w_{|\alpha\beta} - \frac{1}{4} b_{\alpha}^{\rho} u_{\rho|\beta} - \frac{1}{4} b_{\beta}^{\rho} u_{\rho|\alpha} + \frac{3}{4} b_{\alpha}^{\rho} u_{\rho|\beta} + \frac{3}{4} b_{\beta}^{\rho} u_{\rho|\alpha} + b_{\alpha|\beta}^{\lambda} u_{\lambda}.$$

Homogenization of this model by Koiter will result in quite different effective stiffnesses. Speaking more precisely, the form $e(\cdot, \cdot)$, which couples the variational equations in local problems, will be different. Moreover, the form $b(\cdot, \cdot)$ will also change. According to the analysis by KOITER [23], the error of the shell theory is not sensitive to addition or subtraction of terms of the type $b_{(\alpha}^e u_{\beta)}$ in the definition of $\rho_{\alpha\beta}$. Our results are, we hope, correct, at least within the framework of the considered version of the Kirchhoff–Love shell model. Thus the effective stiffnesses characterize the properties of the shell as well as they characterize the model of the shell.

This paper does not close the problems but, we hope, opens the field for further research. For instance, it is a challenging task to examine the effect of coupling of the constitutive relations (3.28) and (3.29) on the errors induced by the homogenized model of the shell.

The asymptotic method of homogenization is not so readily applicable to more complicated shell models of not necessarily the Kirchhoff–Love type. On the other hand, the method of Γ -convergence, and particularly of epi-convergence [4], is applicable to a broad class of linear and nonlinear shells, not necessarily elastic or obeying the Kirchhoff–Love hypothesis, see [40].

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