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Microscopic foundations of the Barkhausen effect

M. SABIR and G. A. MAUGIN (PARIS)

THE THERMODYNAMICAL, phenomenological approach to coupled magnetomechanical hysteresis effects recently proposed by the authors with a view to justifying the use of the Barkhausen effect for measuring states of residual stresses is reviewed and the strong analogies between the rate-independent elastoplasticity with hardening and the magnetic part of the modelling are emphasized. In addition, a semi-microscopic model of magnetic hysteresis accounting for the irreversible motion of magnetic domain walls is presented. Here also a mechanical analogy with the micromechanics of dislocations and crystal plasticity are used whenever possible. However, the natural outcome of this modelling, after passing to the scale of the sample, resembles visco-plasticity in that a time characteristic is involved (the mean transit time of domain walls between the successive structural defects on which they anchor). In the limit of vanishing transit time the above, thermodynamically described, rate-independent behaviour is recovered.

W pracy przedstawiono termodynamiczne i fenomenologiczne podejście do sprzężonych efektów histerezy magnetomechanicznej zaproponowane ostatnio przez autorów w celu wykorzystania efektu Barkhausena do mierzenia stanów naprężeń resztkowych. Podkreślono silne analogie między niezależną od prędkości elastoplastycznością ze wzmocnieniem i magnetyczną częścią modelowania. Ponadto zaprezentowano półmikroskopowy model histerezy magnetycznej uwzględniającej nieodwracalny ruch ścian domen magnetycznych. W przypadkach, gdzie to było możliwe, zastosowano mechaniczne analogie z mikromechaniką dyslokacji i plastycznością kryształu. Jednakże przez wprowadzenie charakterystycznego czasu (średni czas przejścia ścian domen między kolejnymi defektami struktury, na których one kotwiczą) naturalny wynik tego modelowania po przejściu do rozmiarów próbki jest podobny do lepkoplastyczności. Gdy czas przejścia dąży do zera, układ wraca do opisanego wyżej, termodynamicznie niezależnego od prędkości zachowania.

В работе представлены термодинамические и феноменологические подходы к сопряженным эффектам магнетомеханического гистерезиса, предложенного в последнее время авторами с целью использования эффекта Баркгаузена для измерения состояний остаточных напряжений. Подчеркнуты сильные аналогии между независящей от скорости эластопластичностью с упрочнением и магнитной частью моделирования. Кроме этого представлена полумикроскопическая модель магнитного гистерезиса, учитывающая необратимое движение стенок магнитных доменов. В случаях, когда это было возможно, применены механические аналогии с микромеханикой дислокаици и с пластичностью кристалла. Несмотря на введение характеристического времени (среднее время перехода стенок доменов между последовательными дефектами структуры, на которых они закреплены), естественный результат этого моделирования, после перехода к размерам образца, аналогичен вязкопластичности. Когда время перехода стремится к нулю, система возвращается к описанному выше термодинамически поведению, независящему от скорости.

1. Introduction

IN RECENT YEARS several techniques based on the use of electro-magnetomechanical couplings have been devised with a view to developing some aspects of *nondestructive testing*, especially for materials exhibiting residual stresses. Among these techniques we may single out the use of the *Barkhausen effect* to measure residual stresses in magnetizable samples

[1]. In previous works [2, 3] we have developed some theoretical phenomenological aspect concerning a possible description and numerical simulations of this effect. From a pure phenomenological point of view the essential problem was a combination of the now well-established theory of deformable ferromagnets [4, 5] — revisited to allow for nonsaturation of the magnetization — and arguments from continuum thermodynamics involving irreversible processes of the mechanical plastic and magnetic hysteresis types through the concept of internal variables. As a matter of fact, it is observed that the actual shape of the magnetic hysteresis loop depends markedly on the level of stresses and, more particularly, on internal stresses [6, 7], hence the necessity of this coupling between these two phenomena. In fact, the magnetic part of this rather complex behaviour was built in complete parallelism with the mechanical part (rate-independent plasticity with hardening) except for the existence of magnetic saturation and of the first magnetization curve, two aspects of ferromagnetism which rarely have mechanical counterparts. Among the variables introduced in the phenomenological magnetic description, there was the volume *residual magnetization* M^R defined as the vectorial difference between the total magnetization and the thermodynamically reversible magnetization M^r, that is:

$$\mathbf{M}^R = \mathbf{M} - \mathbf{M}^r. \tag{1.1}$$

This relation is analogous to the one for small strains in an elastoplastic body, i.e.

$$\boldsymbol{\epsilon}^p = \boldsymbol{\epsilon} - \boldsymbol{\epsilon}^e, \tag{1.2}$$

where $\boldsymbol{\epsilon}^e$ and $\boldsymbol{\epsilon}^p$ are the elastic and plastic contributions to the total strain $\boldsymbol{\epsilon}$. Elements of the phenomenological theory of stressed ferromagnets exhibiting magnetic hysteresis will be recalled in Sect. 2. The purpose of the present contribution is to provide a semi-microscopic justification of this macroscopic magnetic behaviour by examining the ferromagnetic sample at the scale of magnetic domain walls and then passing to the macroscopic level. Obviously, it is already known that the irreversible magnetization of a sample results from a succession of jerky motions (so-called Barkhausen jumps) of the many domain walls (of various types) in the sample. These magnetic domain walls are anchored on structural defects and there is a "flow" (motion) of a domain wall only if enough magnetic dipole energy is injected in the system to overcome the anchoring energy. We briefly introduce this type of model for *one* magnetic domain wall in Sect. 3 and pass to the resulting macroscopic state of magnetization, or rather the time evolution of this state, in Sect. 4. Although the mechanical analogy continues to exist as in the pure continuum approach, and this is used whenever possible at the different steps of the semi-microscopic modelling, the best analogy which can be drawn is the one with a viscoplastic body (say, Bingham's fluid) in Sect. 5 since a characteristic time is necessarily introduced, contrary to rate-independent effects (such as rate-independent plasticity) which served as models in the thermodynamical theory of Refs. [2] and [3].

2. Phenomenological thermodynamical theory

In this reminder we focus our attention on the "magnetic part" of the model and the influence of mechanical effects such as stresses on it. The main elements of the continuum thermodynamical model of magnetic hysteresis built in Refs. [2, 3] are as follows on account of the additive decomposition (1.1):

Dissipation inequality:

$$\mathbf{H}\cdot\dot{\mathbf{M}}^R-\mathscr{H}\dot{w}\geqslant 0. \tag{2.1}$$

Free energy and state laws:

$$\Psi=\Psi_m(\mathbf{M}^r,\ \cdot)+\Psi_{(i)}(w) \tag{2.2}$$

and

$$\mathbf{H}=\partial\Psi_m/\partial\mathbf{M}^r,\qquad \mathscr{H}=\partial\Psi_{(i)}/\partial w. \tag{2.3}$$

Normality rule (evolution equations):

$$\dot{\mathbf{M}}^R=\dot{\lambda}\frac{\partial f}{\partial\mathbf{H}},\qquad \dot{w}=-\dot{\lambda}\frac{\partial f}{\partial\mathscr{H}}, \tag{2.4}$$

with $\dot{\lambda}\geqslant 0$ if $f=0$ *and* $\dot{f}=0$ and $\dot{\lambda}=0$ if $f<0$ and $\dot{f}<0$. In the above set of equations $\mathbf{H}$ is the magnetic field vector, Ψ is the free energy (thermal influences are ignored; the missing argument in Ψ_m may be a strain), $\dot{\lambda}$ is a multiplier (analogous to the plastic multiplier of elastoplasticity), w is a *scalar* magnetic internal variable, $\mathscr{H}$ (with the physical dimension of a magnetic field) is its conjugated force, and f is the magnetic loading function. The latter defines a convex set, $C=\{\mathbf{H},\mathscr{H}|f(\mathbf{H},\mathscr{H})\leqslant 0\}$ in the three-dimensional space of magnetic fields. The following two results hold true [2, 3]:

$$\dot{\mathbf{H}}\cdot\dot{\mathbf{M}}^R-\dot{\mathscr{H}}\dot{w}=0 \tag{2.5}$$

and

$$\dot{\mathbf{H}}\cdot\dot{\mathbf{M}}^R\geqslant 0, \tag{2.6}$$

of which the first is an orthogonality relation between time rates and the second expresses the magnetic equivalent of "Drucker's postulate" for plasticity with hardening. The latter relation holds good only if the "stocked" magnetic energy $\Psi_{(i)}$ is concave in its argument w. The modelling is completed by the data of the magnetic loading function $f(\mathbf{H},\mathscr{H})$ and the equation of the first magnetization curve $\mathbf{H}=\varphi(\mathbf{M}^R)$ that defines how one starts from a virgin state and tends toward magnetic saturation for large intensities of $\mathbf{H}$:

Example of magnetic loading function (by analogy with the Hubert-Mises plasticity criterion)

$$f(\mathbf{H},\mathscr{H})=(||\mathbf{H}||_a-\mathscr{H})^2-H_c^2,\qquad ||\mathbf{H}||_a=(H_i a_{ij} H_j)^{1/2} \tag{2.7}$$

or the two branches

$$f^{\pm}=(||\mathbf{H}||_a-\mathscr{H})\mp H_c, \tag{2.8}$$

where H_c is the so-called coercive field and a_{ij} accounts for magnetic anisotropy.

First magnetization curve (this is the result of another modelling involving the motion of domain walls that shall be given in a further work):

$$||\mathbf{H}||_a=\varphi(M^R),\qquad M^R=||\mathbf{M}^R||_{a^{-1}} \tag{2.9}$$

such that

$$\varphi(M^R) = -\varphi(-M^R), \quad \varphi^{-1}(\|\mathbf{H}\|_a \to \pm\infty) = \pm M_s^R, \tag{2.10}$$

where M_s^R is the saturation value of the residual magnetization.

It follows from Eq. (2.7) that Eqs. (2.4) are consistent if and only if

$$w(t) = \int_0^t (\dot{M}_i^R(\tau) a_{ij}^{-1} \dot{M}_j^R(\tau)) d\tau \stackrel{\text{def}}{=} \overline{\overline{M^R(t)}} \tag{2.11}$$

which is the so-called *cumulated* residual magnetization in strict analogy with Odqvist's parameter of plasticity with hardening. Thus we have a model that reproduces the formation of magnetic hysteresis loops [3] and includes the effects of transients from a virgin state, magnetic hardening and magnetic saturation. Stresses can also be accounted for in the following manner.

Influence of stresses on the magnetic hysteresis loop

Since the formation of the final magnetic hysteresis loop (so-called major loop) depends on both the function f and φ, the influence of stresses $\boldsymbol{\sigma}$ must be felt through these two functions, i.e. we should write $f = f(\mathbf{H}, \mathscr{H}, \boldsymbol{\sigma})$ and $\varphi = \varphi(\mathbf{M}^R, \boldsymbol{\sigma})$. For f a sensible generalization consists in replacing $\mathbf{H}$ by an effective magnetic field which accounts for the coupling with stresses through a term of the *magnetostriction type* (piezomagnetism is not envisaged

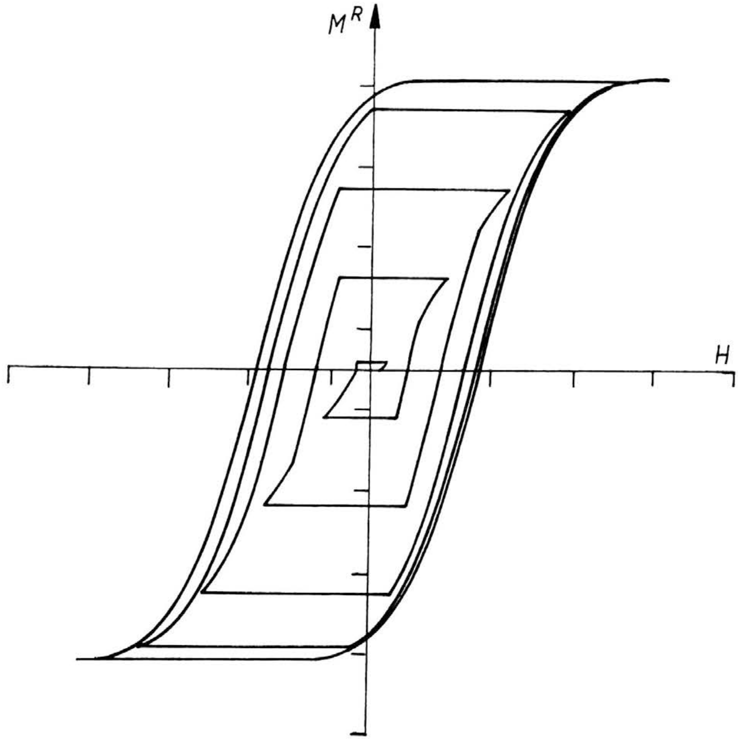

FIG. 1. Formation of the major magnetic hysteresis loop from a virgin state via an alternating magnetic loading.

being seldom met and altogether forbidden in isotropic and cubic structures of interest). For instance, the components H_i of $\mathbf{H}$ are replaced by

$$\hat{H}_i = H_i - 2B_{pqij} S_{pqmn} M_j^R \sigma_{mn}, \tag{2.12}$$

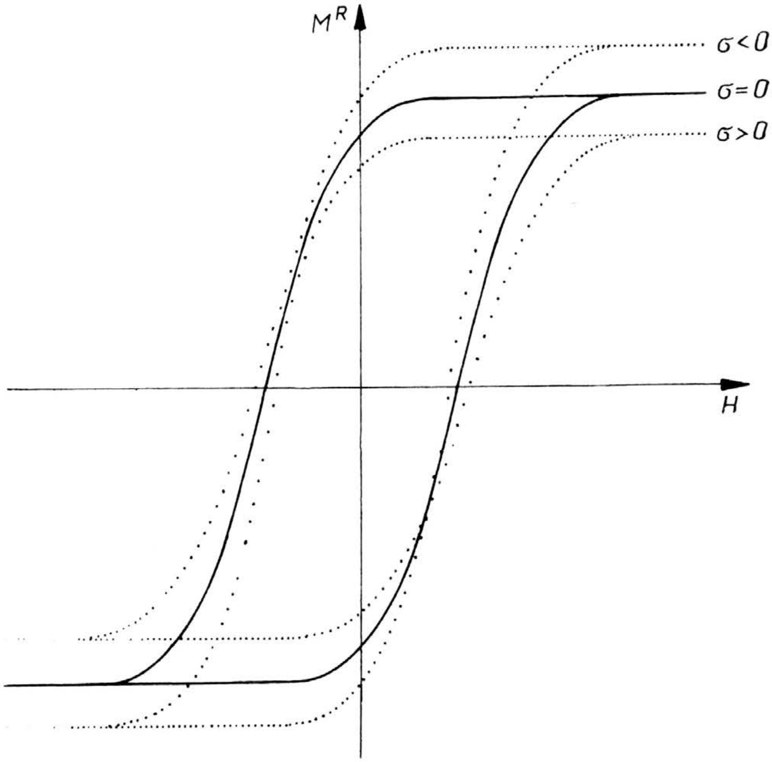

FIG. 2. Qualitative influence of uniaxial stresses on the major magnetic hysteresis loop.

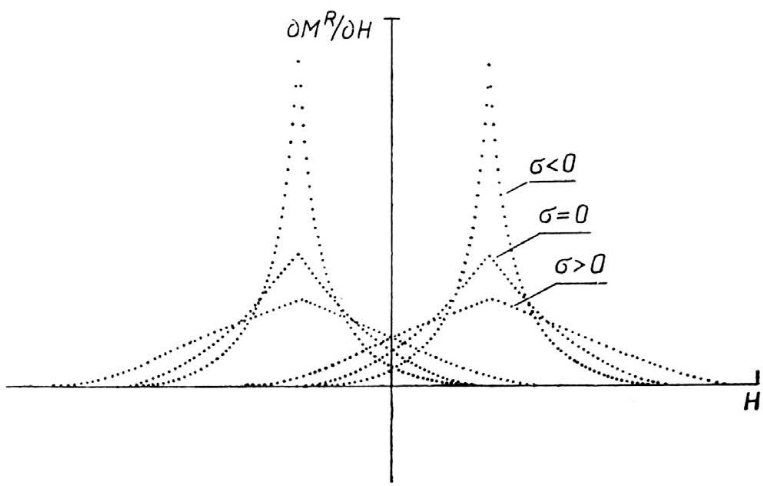

FIG. 3. Qualitative influence of uniaxial stresses on the instantaneous magnetic susceptibility for residual magnetization.

where S_{pqmn} is the tensor of elastic compliances and B_{pqij} is a tensor of "magnetostriction" coefficients. For the one-dimensional model (obtained by projection — indicated below by the subscipt (d) — along a direction $\mathbf{d}$) the influence of stresses on φ is felt through the limit (saturation) condition $(2.10)_2$ by writing an equation of the type

$$M_s^R = M_s^R(\boldsymbol{\sigma} = \mathbf{0})\left(1 - K_{(d)} H^s_{(d)} \sigma_{(d)}\right). \tag{2.13}$$

where $K_{(d)}$ may be either positive or negative, depending on the material. This concludes the reminder on the phenomenological thermodynamical model in which — this must be emphasized — there appears no characteristic time, and which qualitatively reproduces correctly the formation of the magnetic hysteresis loop (Fig. 1) and the influence of stresses on the major hysteresis loop and the instantaneous magnetic susceptibility for residual magnetization (Fig. 2 and 3).

3. Magnetization by motion of magnetic domain walls

3.1. Mean free path of a domain wall

We now consider a totally different avenue based on considerations of the individual motion of magnetic domain walls. This motion is relatively easy in the absence of obstacles. With such obstacles present, impurities or defects, the motion becomes more difficult since it is hindered by a field of pinning pressure (the analogy for domain walls of the Peierls–Nabarro force for dislocations). The most characteristic parameter of the motion of a domain wall is the mean free path. Following an evaluation by L. NÉEL [8], on a line segment of length L on which N_d obstacles are randomly distributed, the mean free path L_w of a wall W is given by

$$L_w = \frac{L}{N_d}\left(1 - \exp(-N_d)\right) \tag{3.1}$$

and this obviously yields $L_w = L$ for $N_d = 0$ and $L_w = L/N_d$ for large N_d.

3.2. Motion of a magnetic domain wall in a region containing obstacles

A ferromagnetic sample is built of many magnetic domains D separated by domain walls W of thickness δ_w (cf. [9]) across which the change in magnetization is supposed to occur through rotation in a smooth manner. To facilitate the analysis, we isolate one domain wall W that separates two magnetic domains D^- and D^+ as indicated in Fig. 4. The wall is schematized mathematically by a discontinuity surface of zero thickness. Let $\mathbf{m}^-$ and $\mathbf{m}^+$ be the magnetic dipole densities on each side of the wall, $\mathbf{m}^-$ and $\mathbf{m}^+$ being spatially uniform in D^- and D^+, respectively, and of equal magnitude. With the convention of hydrodynamical discontinuities, the jump of m across W is defined as

$$[\![\mathbf{m}]\!] = \mathbf{m}^+ - \mathbf{m}^-. \tag{3.2}$$

The moments $\mathbf{m}^+$ and $\mathbf{m}^-$, in general, may have any orientation relative to W. To simplify the presentation, however, we assume that they have no components in the z-direction and this will clearly apply satisfactorily to structures such as a thin film or a whisker with

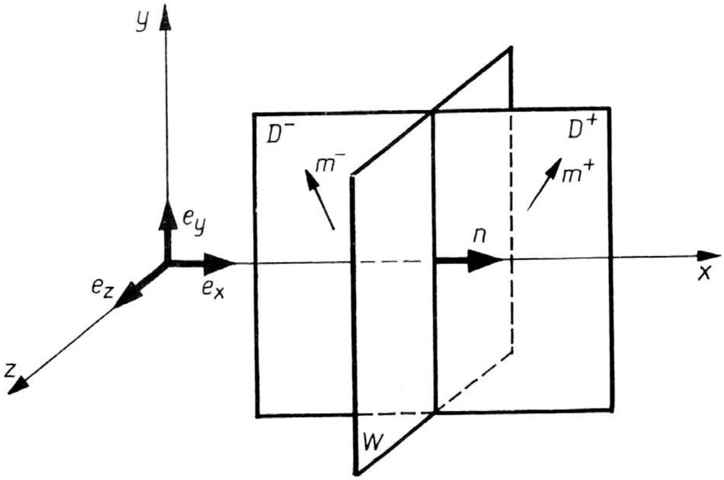

FIG. 4. Magnetic domain wall in motion.

a plane parallel to the $z = 0$-plane. Ultimately the walls will belong to two families of walls, the so-called 180° walls and 90° walls and this will fix the orientation of moments in adjacent domains. When a particular wall W moves by an elementary distance Δx toward positive x (along the **n**-direction) the increase in magnetization due to this displacement is proportional to the area S swept by the wall and we can write in an obvious manner

$$\Delta \mathbf{m} = -[\![\mathbf{m}]\!] S \Delta x. \tag{3.3}$$

$$\dot{\mathbf{m}}^R = -[\![\mathbf{m}]\!] S v_w, \tag{3.4}$$

where $v_w = \mathbf{v} \cdot \mathbf{n}$ is the normal speed of displacement of the wall, **v** being its velocity. Equation (3.4), transcribed per unit volume, is the magnetic analog of the celebrated formula of OROWAN [10] that relates the rate of plastic strain $\dot{\varepsilon}^p$ to the density of mobile dislocations ϱ_m, the (scalar) Burgers vector b (a discontinuity in the elastic displacement) and the mean velocity v_D of dislocations by

$$\dot{\varepsilon}^o = \varrho_m b v_D. \tag{3.5}$$

For a one-dimensional model the speed v_w appearing in Eq. (3.4) can be estimated in the mean as

$$v_w = \frac{L_w}{T_t}, \tag{3.6}$$

where L_w is the mean free path (3.1) and T_t is the mean transit time of a wall between two rest points. T_t is zero if the motion is instantaneous and finite and non-zero when some "viscosity" (obstacles) slows down the wall motion. An increase $\dot{\mathbf{m}}^R$ in magnetization will occur only if a magnetic field, **h** at the microscopic scale, is applied. The magnetic dipole energy in general is defined by $E_m = -\mathbf{m} \cdot \mathbf{h}$. The difference in such an energy on the two faces of the domain wall W acts as a *motive pressure* for the motion of the wall. Thus in agreement with L. NÉEL [8] this pressure p_h (W) is introduced by

$$p_h(W) = [\![E_m]\!] = -[\![\mathbf{m}]\!] \cdot \mathbf{h} \tag{3.7}$$

since $\mathbf{h}$ is the same on both sides of W. But impurities and structural defects have for effect to generate inhomogeneities at a small scale and these, in turn, create antagonist "pressures" which oppose the motive pressure in the course of the wall motion. As a simple model we accept to represent this antagonist, pinning (anchoring) pressure field by an expression formally similar to Eq. (3.7) but with $\mathbf{h}$ replaced by an inhomogeneous pinning or anchoring "magnetic" field $\mathbf{h}_A(\mathbf{x})$, that is, we write

$$p_A(W, \mathbf{x}) = -[\![\mathbf{m}]\!] \cdot \mathbf{h}_A(\mathbf{x}). \tag{3.8}$$

By analogy with the mean time that is necessary for a dislocation to overcome an obstacle (compare ZARKA [11]), we propose that T_t — which depends on both p_h and p_A — be the larger, the closer p_h is to p_A from above (i.e. T_t is inversely proportional to the gap between the motive pressure (3.7) and the anchoring energy (3.8) on the condition that the former be greater than the latter, otherwise there is no motion and thus $\dot{\mathbf{m}}^R = \mathbf{0}$). Hence we propose the expression

$$T_t = \frac{C_0}{\langle p_h - p_A \rangle}, \tag{3.9}$$

where C_0 is a characteristic parameter of the magnetic material and we adopt the convention that

$$\langle A \rangle = \begin{cases} A & \text{if} \quad A > 0, \\ 0 & \text{if} \quad A \leqslant 0. \end{cases} \tag{3.10}$$

Notice that the combination of Eqs. (3.6) and (3.9) provides a "law of motion" $v_w = L_w \langle p_h - p_A \rangle / C_0$ for the domain wall. As a consequence we can rewrite Eq. (3.4) as

$$\dot{\mathbf{m}}^R = \frac{S L_w}{C_0} \left\langle 1 - \frac{||\mathbf{h}_A||}{||\mathbf{h}||} \right\rangle ([\![\mathbf{m}]\!] \otimes [\![\mathbf{m}]\!]) \cdot \mathbf{h}. \tag{3.11}$$

This evolution equation for the residual magnetization related to *one* domain wall clearly is of the "viscoplastic" type since it involves a characteristic time (through C_0) and a threshold (defined by $||\mathbf{h}|| = ||\mathbf{h}_A||$). An interesting limit case is the one where C_0 goes to zero while L_w remains finite, although small (see this limit procedure below for the macroscopic magnetization).

In general domain walls belong to one of the two families known as 180° walls for which $[\![\mathbf{m}]\!] = -2m\mathbf{e}_y$ and 90° walls for which $[\![\mathbf{m}]\!] = -\sqrt{2}m\mathbf{e}_x = -\sqrt{2}m\mathbf{n}$ in the wall frame, if $m = |\mathbf{m}^+| = |\mathbf{m}^-|$. Equation (3.11) thus reads

$$\dot{\mathbf{m}}_i^R = \frac{2 S L_w m^2}{C_0} \left\langle 1 - \frac{||\mathbf{h}_A||}{||\mathbf{h}||} \right\rangle a_{ij} h_j, \tag{3.12}$$

where a_{ij} is a matrix defined in the frame of the wall W, and is such that

$$a_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } 180^\circ \text{ walls}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } 90^\circ \text{ walls}, \tag{3.13}$$

3.3. "Viscoplastic" potential for one domain wall

We pursue the analogy with the micromechanics of dislocations and crystal plasticity processes. Let φ_w $(\mathbf{h}, \theta, w_1, \ldots, w_N)$ be the "viscoplastic" potential of a wall W, where θ is the temperature (in fact assumed constant) and w_α are internal variables. The wall will be in motion only if the following inequality is satisfied:

$$\varphi_w(\mathbf{h}, \theta, w_\alpha) \geqslant 0. \tag{3.14}$$

This is some kind of *activation criterion.* In fact, in analogy with *Schmid's law* of microplasticity [12], we suggest to take for φ_w the expression

$$\varphi_w = \varphi_1(\mathbf{h}, \theta) - \varphi_2(\theta, w_\alpha) \geqslant 0, \tag{3.15}$$

where φ_2 is the limit threshold for the displacement of a wall (this may be zero). In the present case we obtain agreement with Eq. (3.11) by taking

$$\varphi_w = \frac{SL_w}{2C_0} \langle p_h - p_A \rangle^2 \tag{3.16}$$

and we check that Eq. (3.11) is nothing but

$$\dot{\mathbf{m}}_i^R = \left(\frac{\partial \varphi_w}{\partial h_i} \right)_{\{w_\alpha\} = \text{const}}. \tag{3.17}$$

In the space of magnetic fields the surfaces with the constant potential $\varphi_w > 0$ are regular, the surface $\varphi_w = 0$ corresponds to the threshold of reversible motion, and the vector $\dot{\mathbf{m}}^R$ is always normal to the equipotential surfaces — see Eq. (3.17). If we consider a virtual magnetic field $\mathbf{h}^*$ inside the region delineated by the actual potential, then, since the magnetic domain already favourably oriented (relatively to $\mathbf{h}$) will grow at the expense of the other one, we have

$$\dot{\mathbf{m}}^R \cdot \mathbf{h} = -Sv_w[\![\mathbf{m}]\!] \cdot \mathbf{h} \geqslant 0, \qquad \dot{\mathbf{m}}^R \cdot \mathbf{h}^* = 0$$

and thus

$$\dot{\mathbf{m}}^R \cdot (\mathbf{h} - \mathbf{h}^*) \geqslant 0, \tag{3.18}$$

which provides a microscopic magnetic analog of the Hill–Mandel principle of *maximal* dissipation.

4. Macroscopic residual magnetization

Let S' be the area swept in the mean by the magnetic domain walls in a sample. The macroscopic magnetic field $\mathbf{H}$ is equal to $\mathbf{h}$ and the same for all walls in the sample. Let $\bar{\mathbf{h}}_A$ be the statistical average of the anchoring field $\mathbf{h}_A$ and M_s the saturation magnetization per unit volume. We first express the local expression (3.12) in a fixed frame (common to all walls) through a rotation operator that relates this frame and the frame of the individual wall. Then we take the average over all walls (i.e. all possible orientations) noting that the time derivative and average commute. In the process that we do not describe in detail, one introduces $||\mathbf{H}_A|| = \bar{\mathbf{h}}_A$ where a superimposed bar indicates the average. One uses the rule $\overline{AB} = \bar{A}\bar{B}$ (certainly not exactly verified) and has to compute the average of the orientation matrix a_{ij} of Eq. (3.12) once the latter has been expressed in the fixed common frame. We keep the same notation a_{ij} for the resulting average up to a scalar coefficient. The resulting macroscopic equation for $\dot{M}_i^R$ is obtained as

$$\dot{M}_i^R = \frac{2S'L_w M_s^R}{C_0}\left\langle 1-\frac{||\mathbf{H}_A||_a}{||\mathbf{H}||_a}\right\rangle a_{ij}H_j, \tag{4.1}$$

where the macroscopic orientation matrix a_{ij} in fact is proportional to unity if the distribution in orientation of 180° walls or 90° walls in the sample is equiprobable. The macroscopic evolution equation (4.1) is also of the "viscoplastic" type. A macroscopic "viscoplastic" potential Φ can be introduced by

$$\Phi(\mathbf{H}) = \frac{S'L_w}{2C_0}\langle p_H - p_A\rangle^2, \tag{4.2}$$

where

$$p_H = \begin{cases} 2M_s||\mathbf{H}||_a & \text{for } 180^\circ \text{ walls,} \\ \sqrt{2}M_s||\mathbf{H}||_a & \text{for } 90^\circ \text{ walls,} \end{cases} \qquad p_A = \begin{cases} 2M_s||\mathbf{H}_A||_a & \text{for } 180^\circ \text{ walls,} \\ \sqrt{2}M_s||\mathbf{H}_A||_a & \text{for } 90^\circ \text{ walls,} \end{cases} \tag{4.3}$$

where the norm of magnetic fields is computed with the aid of a_{ij} in agreement with the definition $(2.7)_2$. It is immediately verified that

$$\dot{\mathbf{M}}^R = \frac{\partial\Phi}{\partial\mathbf{H}}. \tag{4.4}$$

The macroscopic maximal-dissipation inequality

$$\dot{\mathbf{M}}^R\cdot(\mathbf{H}-\mathbf{H}^*) \geqslant 0 \tag{4.5}$$

can be shown to hold for any $\mathbf{H}^*$ inside the region of magnetic-field space delimited by the actual equipotential surface $\Phi = \text{const} > 0$.

We rewrite Eq. (4.1) as

$$\dot{\mathbf{M}}^R = \frac{1}{C_1}\left\langle 1-\frac{||\mathbf{H}_A||_a}{||\mathbf{H}||_a}\right\rangle \mathbf{a}\cdot\mathbf{H} = \frac{\partial\Phi}{\partial\mathbf{H}}. \tag{4.6}$$

This is the form to be compared with the "plastic" type of magnetization incremental law $(2.4)_1$ which, on account of couplings with stresses and of magnetic hardening, reads

$$\dot{\mathbf{M}}^R = \dot{\lambda}\frac{\partial}{\partial\mathbf{H}}f(\mathbf{H}, \mathscr{H}, \boldsymbol{\sigma}). \tag{4.7}$$

The effect of residual stresses $\boldsymbol{\sigma}$ in Eq. (4.6) can only be felt through $\mathbf{H}_A$ since this field accounts in some way for the distribution of structural defects. The relation between $\mathbf{H}_A$ and the field σ is presently under study. In the thermodynamical continuum context where the magnetic loading function is chosen as (2.7), it is found — for $\boldsymbol{\sigma} = 0$ — that Eq. (4.7) yields

$$\dot{M}_i^R = \overline{\overline{\dot{M}^R(t)}}\frac{a_{ij}H_j}{||\mathbf{H}||_a} \quad \text{at} \quad f^\pm = 0, \tag{4.8}$$

where $\overline{\overline{M^R}}$ is the cumulated residual magnetization. We can compare Eqs. (4.6) and (4.8) when C_1 goes to zero (no characteristic time).

5. Analogy between the magnetic behaviour and Bingham's fluid

The evolution equation (4.6) can be deduced from a dissipation potential $\mathcal{D}$ as follows (compare [3]). The magnetic dissipation reads

$$\Phi_m = \mathbf{H} \cdot \dot{\mathbf{M}}^R \geqslant 0 \tag{5.1}$$

in the absence of internal variables. Consider a special case for which

$$\mathcal{D} = ||\mathbf{H}_A||_{a^{-1}} \cdot ||\dot{\mathbf{M}}^R|| + C_1 ||\dot{\mathbf{M}}^R||^2_{a^{-1}}, \tag{5.2}$$

where the first contribution is strictly analogous to the dissipated power of elastoplasticity (the critical stress level σ_c and the plastic strain rate $\dot{\boldsymbol{\epsilon}}^p$ playing the roles of $||\mathbf{H}_A||$ and $\dot{\mathbf{M}}^R$ respectively), being homogeneous of degree one in $\dot{\mathbf{M}}^R$, while the second contribution is homogeneous of degree two (in fact quadratic) in $\dot{\mathbf{M}}^R$ and therefore resembles the Rayleigh dissipation potential of viscous fluids (with C_1 playing the role of viscosity). Using Euler's identity for homogeneous functions of degrees one and two, one deduces from Eq. (5.2) the equation for $\mathbf{H}$ in terms of $\dot{\mathbf{M}}^R$ and by inversion this yields Eq. (4.6) if $||\mathbf{H}||_a \geqslant ||\mathbf{H}_A||_a$ and zero if $||\mathbf{H}||_a < ||\mathbf{H}_A||_a$. This situation is entirely analogous to that describing the flow of a Bingham fluid ([13], pp. 228–230) if one makes the correct identification (see Table 1).

Table 1. Analogy between residual magnetization process and flow of a Bingham fluid

Residual magnetization	Bingham fluid
$\dot{\mathbf{M}}^R$: rate	D: strain rate
C_1	2μ: viscosity
$\|\|\mathbf{H}_A\|\|$: anchoring field	g: threshold
$\|\|\mathbf{H}\|\|$: norm of magnetic field	$\sqrt{\sigma_{II}}$: (second invariant of stress)$^{1/2}$
$\mathbf{H}$: magnetic field	$\boldsymbol{\sigma}^D$: deviator of stresses

If C_1 tends to zero, we are left with the first contribution in Eq. (5.2) and the equation for $\mathbf{H}$ simply reads

$$\mathbf{H} = \frac{\partial \mathcal{D}}{\partial \dot{\mathbf{M}}^R} \tag{5.3}$$

or

$$H_i = a^{-1}_{ij} \dot{M}^R_j \frac{||\mathbf{H}_A||_a}{||\dot{\mathbf{M}}^R||_{a^{-1}}}. \tag{5.4}$$

The latter equation expresses the fact that $||\mathbf{H}||_a = ||\mathbf{H}_A||_a$ and the vector $\dot{\mathbf{M}}^R$ is proportional to $\mathbf{a} \cdot \mathbf{H}$ so that, introducing a "plastic multiplier" $\dot{\lambda}$, one can write a priori

$$\dot{\mathbf{M}}^R_i = 2\dot{\lambda} a_{ij} H_j \frac{||\mathbf{H}_A||_a}{||\mathbf{H}||_a} \tag{5.5}$$

with $\dot{\lambda} = 0$ if $||\mathbf{H}||_a < ||\mathbf{H}_A||_a$ and $\lambda > 0$ if $||\mathbf{H}||_a = ||\mathbf{H}_A||_a$, in which case Eq. (5.5) reduces to

$$\dot{M}_i^R = 2\dot{\lambda} a_{ij} H_j \tag{5.6}$$

and

$$\dot{\lambda} = ||\dot{\mathbf{M}}^R||_{a^{-1}}/2||\mathbf{H}_A||_a. \tag{5.7}$$

If we set

$$\dot{w} = ||\dot{\mathbf{M}}^R||_{a^{-1}}, \tag{5.8}$$

then $w = \overline{\overline{M^R(t)}}$ in agreement with the definition $(2.11)_2$ and thus Eq. (4.6) goes into Eq. (4.8) in the limit of vanishing C_1 or vanishing transit time, a situation for which the magnetization of the sample occurs through instantaneous jumps of a statistic population of magnetic domain walls. This completes the set of analogies between macroscopic elasto-plasticity with hardening and rate-independent magnetic hysteresis on the one hand, and the micromechanics of dislocations and the jerky motion of magnetic domain walls on the other, together with the passing from the microscopic level to the thermodynamically governed macroscopic level. A theory of ferroelectric hysteresis can be built along the same line by using the same kinds of analogies [14] although the underlying microscopic mechanisms may be quite different.

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UNIVERSITE PIERRE-ET-MARIE CURIE
LABORATOIRE DE MODELISATION EN MECANIQUE, PARIS, FRANCE.

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