

Localization problem of nonlocal continuum theories(*)

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THE AIM of the paper is to explore the primary physical consequences of the biadditivity and continuity assumptions in the nonlocal continuum theory. We investigate the functions appearing in the scalar balance equation. It is shown that the usual continuity assumption is not sufficient for the flux to be separable into surface and volume parts. On the other hand we prove that both continuity and biadditivity constitute the sufficient conditions of uniqueness of such a decomposition. Finally we show the counterpart of the classical Cauchy's theorem for the surface flux in the nonlocal continuum.

Celem pracy jest zbadanie pierwotnych konsekwencji fizycznych, wynikających z założenia biaddytywności i ciągłości nielokalnej teorii ośrodka ciągłego. Badamy funkcje pojawiające się w skalarnym równaniu bilansu. Wykazuje się, że zwykłe założenie ciągłości nie wystarcza do podziału strumienia na część powierzchniową i objętościową. Z drugiej strony dowodzi się, że biaddytywność i ciągłość stanowią warunki wystarczające dla jednoznaczności tego podziału. Wreszcie wykazujemy dla nielokalnego ośrodka ciągłego odpowiednik klasycznego twierdzenia Cauchy'ego dla strumienia powierzchniowego.

Целью работы является исследование первичных физических последствий вытекающих из предположения о биaddyтивности и непрерывности в нелокальной теории сплошной среды. Исследуются функции входящие в скалярное уравнение баланса. Доказывается, что обычные предположения непрерывности недостаточны для разделения потока на поверхностную и объемную части. С другой стороны доказывается, что биaddyтивность и непрерывность являются достаточными условиями для однозначности этого разделения. Наконец, для нелокальной сплошной среды доказывается аналог классической теоремы Коши для поверхностного потока.

1. Preliminary remarks

IN THE CONSTRUCTION of a continuous model of physical systems we may proceed in at least two ways:

i) We may adopt the viewpoint of classical mechanics and assume that the material point is the primitive concept of the theory; in such a case we assume simultaneously the existence of such fields as the mass density, momentum density, specific internal energy etc. Moreover, assuming a so-called principle of local action we are able to determine such quantities as the momentum and internal energy of any collection of material points, possessing the non-zero volume measure, by performing spatial integration of the appropriate fields. If the spatially nonlocal interactions are taken into account, the above mentioned quantities will not be well-defined. Hence, this approach does not lead, in general, to the well-defined notion of a subbody. It is a serious disadvantage of the method because it cannot describe any measurements in the system. In the continuous model, meas-

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uring certain quantities which characterize the body means the creation of a sum of the body and a system which is assumed to be a measuring device; however this operation cannot be defined within the framework of the presented model.

On the other hand, the model possesses the advantage of being described by the point functions, what makes possible at least formally, the effective solutions of the boundary value problems.

ii) We may construct the continuous model basing on the notion of a subbody. This approach leads to such notions as the mass, momentum and internal energy of an arbitrary subbody and it clearly states the continuity assumptions. The notion of a material point in such a theory is not, in general, well-defined. If we assume again the principle of local action, both approaches become identical.

The main advantage of the model described above is the possibility of measuring the characteristic quantities through the utilization of systems considered to be measuring devices. At the same time this model seems to reflect all features usually connected with the continuum theories.

However, in the general case of spatially nonlocal interactions, the set functions describing the model cannot be represented by the point functions, which define the fields similar to these of the first approach. The question of the existence of such representations is called the localization problem. It is of extreme importance for the effectiveness of this approach. There seems to be no other way of posing, for instance, the boundary value problems.

The first type of models was extensively treated by D.G.B. EDELEN [1976] where other references can be found. The second type of models was proposed by M. E. GURTIN and W. O. WILLIAMS [1967] who solved the simplest problem of localization. A review of papers on this subject can be found, for example, in my monograph [1974].

In the present work we deal with the continuous model of the second type. The main aim of the paper is to show systematically the transition from the general model to certain localizable problems. We present the procedure in the case of the scalar balance equation but it can be easily generalized to other cases. As we see further, the main cause of difficulties is the presence of nonlocal interactions. Hence we should first give reasons for the necessity of their appearance. It seems to me that we can distinguish two physical types of nonlocalities in the configuration space. The first one is connected with the internal structure of the material, the dimensions of the sample (size effects) etc. Such nonlocalities depend weakly on the rate of the process which takes place in the material. The second type depends strongly on the rate of the process and the intensity of interactions increases with this rate. Thermal internal radiation is an example of such a nonlocality.

Roughly speaking, the non-locality is the price we pay for fields participating in the process but neglected in the formal model. The simplest theory where such an effect can be observed is the thermomechanics of materials applied to describe shock waves of a very high amplitude. Such waves yield very often local ionization of the medium and, consequently, generation of secondary electromagnetic waves. The latter can be accounted for either by the coupled thermomechanical-electromagnetic field model or, approximately, by a spatially nonlocal purely thermomechanical model. The nonlocality of the latter follows from electrostrictive effects advancing in the configuration space all thermomechan-

ical impulses. It is schematically shown in Fig. 1. The initial source (i) generates a thermo-mechanical shock wave (tmw) of the amplitude high enough to ionize the medium in the vicinity of the front (shw). Then the front (shw) becomes a collection of secondary sources (s) of electromagnetic waves (emw), changing the temperature and the deformation of the medium at points X in front of the shock wave. The last phenomenon is due to the electro-

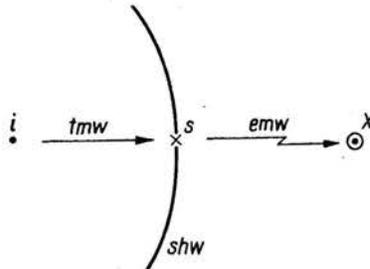


FIG. 1.

striction of the material. In the local thermomechanical theory such phenomena are forbidden. The natural method of extension of this theory is to introduce nonlocal thermo-mechanical interactions. It is obvious that in the case of known fields carrying "nonlocal" impulses, it is better to use a proper coupled theory rather than to introduce spatial non-localities. We have such a case in the example presented above. However, many physical phenomena cannot be explained by the coupling of contemporarily known fields either due to the complexity of coupled theories or due to the fact that this is different from all known couplings. We face such a problem, for instance, in the elementary particle theory of quark interactions.

Similar difficulties were predicted by statistical theories where the high deviation from thermodynamic equilibrium yields an increment of the number of independent variables describing the state of the system. It means that the process in such a system is usually not a curve in the phase space of fixed dimension.

The nonlocal theories seem to solve the problem of the existence of impulse carriers, different from those allowed in the considered theory and changing with the deviation from the equilibrium. This approach fits well in the pattern of phenomenological theories and, at least in the case of thermodynamic theories of materials seems to reflect the experimental evidence with sufficient accuracy. We return to the latter problem in more specialized papers.

2. Scalar balance equation

We limit our considerations to systems which are collections of points of the bounded subset of the three-dimensional Euclidean space E^3 . The geometry of such a system is characterized by the structure of subsystems. We assume this structure to be similar to that introduced by M. E. GURTIN and W. O. WILLIAMS [1967]. Namely, we consider the structure, a Boolean algebra B , to be described by the following relations:

$$\begin{aligned}
 \forall_{\mathcal{P}_1, \mathcal{P}_2 \in \mathbf{B}} \mathcal{P}_1 \vee \mathcal{P}_2 &:= \mathcal{P}_1 \cup \mathcal{P}_2, & \mathcal{P}_1 \wedge \mathcal{P}_2 &:= \overline{\text{int}(\mathcal{P}_1 \cap \mathcal{P}_2)}, & \mathcal{P}_1 < \mathcal{P}_2 &\Leftrightarrow \mathcal{P}_1 \wedge \mathcal{P}_2 = \mathcal{P}_1, \\
 (2.1) \quad \exists_{\phi \in \mathbf{B}} \forall_{\mathcal{P} \in \mathbf{B}} \phi \vee \mathcal{P} &= \mathcal{P}, & \exists_{\mathcal{S} \in \mathbf{B}} \forall_{\mathcal{P} \in \mathbf{B}} \mathcal{S} \wedge \mathcal{P} &= \mathcal{P}, \\
 \forall_{\mathcal{P} \in \mathbf{B}} \exists_{\mathcal{P}^e \in \mathbf{B}} \mathcal{P}^e &:= \overline{\mathcal{P} - \mathcal{P}},
 \end{aligned}$$

where int and $\overline{(\cdot)}$ are the interior and closure, respectively, relative to the topology of E^3 . The members of \mathbf{B} are regular subsets of E^3 , i.e. they are closures of their interiors and their boundaries are finite unions of C^2 — manifolds of the dimension 2 or less. For any boundary $\partial \mathcal{P}_1$ of $\mathcal{P}_1 \in \mathbf{B}$, we assume that there is a sequence $\{\mathcal{A}_i\}_{i=1}^\infty$ such that

$$(2.2) \quad \forall_{\mathcal{S} \in \mathbf{B}} \mathcal{S} := \overline{\text{int}(\partial \mathcal{P}_1 \cap \mathcal{P}_2)} = \bigcap_{i=1}^\infty \mathcal{A}_i,$$

where int and $\overline{(\cdot)}$ are the interior and closure in the topology of $\partial \mathcal{P}_1$. Each \mathcal{S} of the form (2.2) is called a material surface. We assume it to have orientation of the exterior normal to \mathcal{P}_1 , i.e.

$$(2.3) \quad \text{orientation} \overline{(\text{int}(\partial \mathcal{P}_1 \cap \mathcal{P}_2))} = -\text{orientation} \overline{(\text{int}(\partial \mathcal{P}_2 \cap \mathcal{P}_1))}.$$

We say that two subsystems \mathcal{P}_1 and \mathcal{P}_2 are non-overlapping if

$$(2.4) \quad \mathcal{P}_1 \wedge \mathcal{P}_2 = \phi.$$

We use the following notation:

$$(2.5) \quad \text{sep} \mathbf{B} \times \mathbf{B} := \{(\mathcal{P}_1, \mathcal{P}_2) \in \mathbf{B} \times \mathbf{B} \mid \mathcal{P}_1 \wedge \mathcal{P}_2 = \phi\}.$$

We assume the existence of the following functions:

$$(2.5') \quad \begin{aligned} E(\cdot; t) &: \mathbf{B} \rightarrow R, \\ Q(\cdot, \cdot; t) &: \text{sep} \mathbf{B} \times \mathbf{B} \rightarrow R \end{aligned}$$

satisfying the balance equation

$$(2.6) \quad \forall_{\mathcal{P} \in \mathbf{B}} \dot{E}(\mathcal{P}; t) = Q(\mathcal{P}, \mathcal{P}^e; t), \quad \dot{E}(\mathcal{P}; t) := \frac{d}{dt} E(\mathcal{P}; t).$$

Simultaneously, we assume the existence of a time derivative of the function $E(\mathcal{P}; \cdot)$ for all instances in the considered interval. The function $Q(\cdot, \cdot; t)$ is called a flux for the state function $E(\cdot; t)$. Further, we do not consider the evolution properties and therefore the time is the parameter. To simplify the printing we drop this argument with the understanding that all considered properties are the same for any instant of time.

In addition to the above assumptions, we require some continuity conditions to be fulfilled. We introduce these requirements in the sequel of the paper as the needs arise. We start with the continuity assumption for the flux.

Let v be the complete Lebesgue volume measure in E^3 and s be the complete Lebesgue surface measure on the two-dimensional manifolds in E^3 . Then

AXIOM:

$$(2.7) \quad \exists_{\alpha, \beta > 0} \forall_{(\mathcal{P}_1, \mathcal{P}_2) \in \text{sep} \mathbf{B} \times \mathbf{B}} |Q(\mathcal{P}_1, \mathcal{P}_2)| \leq \alpha v(\mathcal{P}_1) v(\mathcal{P}_2) + \beta s(\partial \mathcal{P}_1 \cap \mathcal{P}_2), \quad v(\mathcal{P}) < +\infty.$$

We assume α, β to be the least Lipschitz constants ●

The above axiom yields a decomposition of the flux into the surface and volume parts, but it is not sufficient for the existence of Lebesgue-Radon-Nikodym derivatives of $Q(\cdot, \cdot)$ with respect to v and s . The existence of such derivatives requires an assumption on the biadditivity of fluxes. We do not make this assumption for the reasons explained in my paper [1977]. Certain aspects of this problem are presented below in Sect. 3.

3. Surface and volume fluxes

Let us consider two non-overlapping subsystems \mathcal{P}_1 and \mathcal{P}_2 such that $\partial\mathcal{P}_1 \cap \mathcal{P}_2 \neq \emptyset$. We introduce the descending family of subsystems $\{\mathcal{A}_i\}_{i=1}^{\infty}$ endowed with the following properties:

$$(3.1) \quad \begin{aligned} \text{i)} \quad & \forall_i \mathcal{A}_i < \mathcal{P}_2, \\ \text{ii)} \quad & \forall_i \partial\mathcal{P}_1 \cap \mathcal{P}_2 = \partial\mathcal{P}_1 \cap \mathcal{A}_i, \\ \text{iii)} \quad & \bigcap_{i=1}^{\infty} \mathcal{A}_i = \partial\mathcal{P}_1 \cap \mathcal{P}_2. \end{aligned}$$

The surface $\partial\mathcal{P}_1 \cap \mathcal{P}_2$ is material and, according to its definition, such a family can be chosen from the algebra \mathcal{B} .

As we have pointed out in the paper [1977], there are at least two decompositions of the flux which follow from the Lipschitz continuity assumption. Assuming the existence of the following limits,

$$(3.2) \quad \begin{aligned} & \lim_{i \rightarrow \infty} Q(\mathcal{P}_1, \mathcal{A}_i), \\ & \lim_{i \rightarrow \infty} Q(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_i}), \end{aligned}$$

we can define the surface flux from the subsystem \mathcal{P}_2 to the subsystem \mathcal{P}_1 as

$$(3.3) \quad Q_s(\mathcal{P}_1, \mathcal{P}_2) := \lim_{i \rightarrow \infty} Q(\mathcal{P}_1, \mathcal{A}_i)$$

or

$$(3.4) \quad \overline{Q}_s(\mathcal{P}_1, \mathcal{P}_2) := Q(\mathcal{P}_1, \mathcal{P}_2) - \lim_{i \rightarrow \infty} Q(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_i}).$$

These definitions are not equivalent for non-additive fluxes $Q(\mathcal{P}_1, \cdot)$. It has been shown on the simple counter-example in the paper [1977]. The problem is even more difficult when one of the limits (3.2) ceases to exist. If neither of them exists, there seems to be no way of defining the surface and volume parts of the flux.

It should be pointed out that the assumption on the existence of the limits (3.2) has not been explicitly made in the paper [1977]. This assumption is not required in the case of biadditive fluxes because the Lipschitz continuity and biadditivity are sufficient for the existence of these limits. Namely,

$$\forall_{\substack{\mathcal{A}_i, \mathcal{A}_j \\ j > i}} Q(\mathcal{P}_1, \mathcal{A}_j) - Q(\mathcal{P}_1, \mathcal{A}_i) = -Q(\mathcal{P}_1, \overline{\mathcal{A}_i - \mathcal{A}_j}).$$

On the other hand

$$\overline{\mathcal{A}_i - \mathcal{A}_j} \cap \mathcal{P}_1 = \phi$$

and, according to the continuity assumption (2.7), we have

$$|Q(\mathcal{P}_1, \overline{\mathcal{A}_i - \mathcal{A}_j})| \leq \alpha v(\mathcal{P}_1) v(\overline{\mathcal{A}_i - \mathcal{A}_j}).$$

Hence

$$\forall \varepsilon > 0 \exists j, i > N \Rightarrow |Q(\mathcal{P}_1, \mathcal{A}_j) - Q(\mathcal{P}_1, \mathcal{A}_i)| < \varepsilon.$$

According to the Bolzano–Cauchy's theorem, the sequence $\{Q(\mathcal{P}_1, \mathcal{A}_i)\}_{i=1}^{\infty}$ is convergent. Simultaneously

$$Q(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_i}) = Q(\mathcal{P}_1, \mathcal{P}_2) - Q(\mathcal{P}_1, \mathcal{A}_i).$$

It means that the sequence $\{Q(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_i})\}_{i=1}^{\infty}$ is also convergent.

However, the Lipschitz continuity assumption (2.7) without the biadditivity of the flux does not suffice for the existence of the limits (3.2) (1).

Let us consider three possible cases which lead to the decomposition of the non-additive flux. Our goal is to find such functions $Q_s(\mathcal{P}_1, \cdot)$ and $Q_v(\mathcal{P}_1, \cdot)$ that

$$(3.5) \quad \forall (\mathcal{P}_1, \mathcal{P}_2) \in \text{sep} \mathbb{B} \times \mathbb{B} \quad Q(\mathcal{P}_1, \mathcal{P}_2) = Q_s(\mathcal{P}_1, \mathcal{P}_2) + Q_v(\mathcal{P}_1, \mathcal{P}_2)$$

and

$$(3.6) \quad \begin{aligned} \exists \tilde{\alpha} > 0 \quad \forall (\mathcal{P}_1, \mathcal{P}_2) \in \text{sep} \mathbb{B} \times \mathbb{B} \quad |Q_v(\mathcal{P}_1, \mathcal{P}_2)| &\leq \tilde{\alpha} v(\mathcal{P}_1) v(\mathcal{P}_2), \\ \exists \tilde{\beta} > 0 \quad \forall (\mathcal{P}_1, \mathcal{P}_2) \in \text{sep} \mathbb{B} \times \mathbb{B} \quad |Q_s(\mathcal{P}_1, \mathcal{P}_2)| &\leq \tilde{\beta} s(\partial \mathcal{P}_1 \cap \mathcal{P}_2). \end{aligned}$$

As we have mentioned, the decomposition (3.5) is unique if the function $Q(\mathcal{P}_1, \cdot)$ is additive. We do not make this assumption, i.e.

$$(3.7) \quad \exists \mathcal{A}_1 \quad Q(\mathcal{P}_1, \mathcal{P}_2) - Q(\mathcal{P}_1, \mathcal{A}_1) - Q(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_1}) \neq 0.$$

We proceed now to the consideration of the case in which both limits (3.2) exist. Then, the decomposition (3.5) and the continuity assumption (3.6) yield at once the following:

LEMMA. If the surface flux $Q_s(\mathcal{P}_1, \cdot)$ and the volume flux $Q_v(\mathcal{P}_1, \cdot)$ satisfying the conditions (3.5) exist, then

$$(3.8) \quad \begin{aligned} \lim_{i \rightarrow \infty} Q(\mathcal{P}_1, \mathcal{A}_i) &= \lim_{i \rightarrow \infty} Q_s(\mathcal{P}_1, \mathcal{A}_i), \\ \lim_{i \rightarrow \infty} Q(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_i}) &= \lim_{i \rightarrow \infty} Q_v(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_i}). \end{aligned}$$

Proof. The former relation follows from the completeness of the volume measure

$$(3.9) \quad \lim_{i \rightarrow \infty} v(\mathcal{A}_i) = 0,$$

(1) The author is very grateful to W. O. WILLIAMS for pointing out this inadvertence of the paper [1977].

while the latter is an obvious consequence of the following relation

$$(3.10) \quad Q(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_i}) = Q_v(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_i}),$$

which holds for any two disjoint subsystems $\mathcal{P}_1, \mathcal{P}_2 - \mathcal{A}_i$ ■

Let us consider the non-additivity of $Q(\mathcal{P}_1, \cdot)$. According to the relation (3.10), we have

$$Q(\mathcal{P}_1, \mathcal{P}_2) - Q(\mathcal{P}_1, \mathcal{A}_i) - Q(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_i}) = Q_s(\mathcal{P}_1, \mathcal{P}_2) + Q_v(\mathcal{P}_1, \mathcal{P}_2) - Q_s(\mathcal{P}_1, \mathcal{A}_i) - Q_v(\mathcal{P}_1, \mathcal{A}_i) - Q_v(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_i}).$$

Shrinking down the family $\{\mathcal{A}_i\}_{i=1}^\infty$ to the surface $\partial\mathcal{P}_1 \cap \mathcal{P}_2$, we arrive at

$$(3.11) \quad \lim_{i \rightarrow \infty} \{Q(\mathcal{P}_1, \mathcal{P}_2) - Q(\mathcal{P}_1, \mathcal{A}_i) - Q(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_i})\} = \{Q_s(\mathcal{P}_1, \mathcal{P}_2) - \lim_{i \rightarrow \infty} Q(\mathcal{P}_1, \mathcal{A}_i)\} + \{Q_v(\mathcal{P}_1, \mathcal{P}_2) - \lim_{i \rightarrow \infty} Q(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_i})\}.$$

Obviously, the biadditivity assumption together with the formula (3.11) leads to the following relations:

$$(3.12) \quad \begin{aligned} Q_s(\mathcal{P}_1, \mathcal{P}_2) &= \lim_{i \rightarrow \infty} Q(\mathcal{P}_1, \mathcal{A}_i), \\ Q_v(\mathcal{P}_1, \mathcal{P}_2) &= \lim_{i \rightarrow \infty} Q(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_i}). \end{aligned}$$

However, in the general case neither of the relations (3.12) holds. Without contradicting the formula (3.11), we can choose the decomposition (3.5) in such a way that one of the relations (3.12) will be satisfied. Let us discuss both possibilities. First, let us assume that

$$(3.13) \quad Q_s(\mathcal{P}_1, \mathcal{P}_2) := \lim_{i \rightarrow \infty} Q(\mathcal{P}_1, \mathcal{A}_i).$$

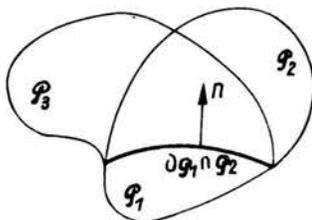


FIG. 2.

Then, for any two subsystems $\mathcal{P}_2, \mathcal{P}_3$ such that (Fig. 2)

$$(3.14) \quad \partial\mathcal{P}_1 \cap \mathcal{P}_2 = \partial\mathcal{P}_1 \cap \mathcal{P}_3 \quad \text{and} \quad \mathcal{P}_1 \wedge \mathcal{P}_2 = \mathcal{P}_1 \wedge \mathcal{P}_3 = \phi$$

we have

LEMMA

$$(3.15) \quad Q_s(\mathcal{P}_1, \mathcal{P}_2) = Q_s(\mathcal{P}_1, \mathcal{P}_3).$$

PROOF. The family $\{\mathcal{A}_i \cap \mathcal{P}_3\}_{i=1}^\infty$ satisfies the same conditions as $\{\mathcal{A}_i\}_{i=1}^\infty$. On the other hand, as we have proved in [1977], the definition (3.13) does not depend on the choice of $\{\mathcal{A}_i\}_{i=1}^\infty$ satisfying Eq. (3.1). Hence

$$Q_s(\mathcal{P}_1, \mathcal{P}_3) := \lim_{i \rightarrow \infty} Q(\mathcal{P}_1, \mathcal{A}_i \cap \mathcal{P}_3) = Q_s(\mathcal{P}_1, \mathcal{P}_2) \blacksquare$$

It means that the definition (3.13) yields the independence of the surface flux $Q_s(\cdot, \cdot)$ on the interaction of \mathcal{P}_1 with $\mathcal{P}_2 - \partial\mathcal{P}_1$. It describes solely the surface properties of the flux $Q(\mathcal{P}_1, \cdot)$, as predicted by physical intuition. Unfortunately, it does not mean that $Q_s(\cdot, \cdot)$ is local or even biadditive. It still contains the interactions among points on the surface and, in general, is not balanced

$$\exists_{(\mathcal{P}_1, \mathcal{P}_2) \in \text{sepB} \times \text{B}} Q_s(\mathcal{P}_1, \mathcal{P}_2) \neq -Q_s(\mathcal{P}_2, \mathcal{P}_1).$$

According to the formula (3.11) and the definition (3.13), we have for non-additive fluxes

$$(3.16) \quad \exists_{(\mathcal{P}_1, \mathcal{P}_2) \in \text{sepB} \times \text{B}} Q_v(\mathcal{P}_1, \mathcal{P}_2) \neq \lim_{i \rightarrow \infty} Q_v(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_i}).$$

The decomposition based on the definition (3.13) satisfies the continuity condition which is described by the following:

THEOREM

$$(3.17) \quad \begin{aligned} & \forall_{(\mathcal{P}_1, \mathcal{P}_2) \in \text{sepB} \times \text{B}} |Q_s(\mathcal{P}_1, \mathcal{P}_2)| \leq \beta s(\partial\mathcal{P}_1 \cap \mathcal{P}_2), \\ & \exists_{\tilde{\alpha} > \alpha} \forall_{(\mathcal{P}_1, \mathcal{P}_2) \in \text{sepB} \times \text{B}} |Q_v(\mathcal{P}_1, \mathcal{P}_2)| \leq \tilde{\alpha} v(\mathcal{P}_1) v(\mathcal{P}_2), \end{aligned}$$

where α, β are the least Lipschitz constants for $Q(\cdot, \cdot)$ (see the formula (2.7)) and $\tilde{\alpha}$ is the least Lipschitz constant for $Q_v(\cdot, \cdot)$.

PROOF. The former inequality follows at once from the definition (3.13) and the continuity of $Q(\cdot, \cdot)$:

$$\lim_{i \rightarrow \infty} |Q(\mathcal{P}_1, \mathcal{A}_i)| \leq \lim_{i \rightarrow \infty} \{ \alpha v(\mathcal{P}_1) v(\mathcal{A}_i) + \beta s(\partial\mathcal{P}_1 \cap \mathcal{A}_i) \} = \beta s(\partial\mathcal{P}_1 \cap \mathcal{P}_2)$$

with respect to the completeness of the Lebesgue volume measure v .

The latter inequality is obvious in the case of two subsystems such that

$$s(\partial\mathcal{P}_1 \cap \mathcal{P}_2) = 0.$$

In such a case, according to the relation (3.10) and the continuity of $Q(\cdot, \cdot)$, we have

$$|Q_v(\mathcal{P}_1, \mathcal{P}_2)| \leq \alpha v(\mathcal{P}_1) v(\mathcal{P}_2).$$

Let us consider the descending family of the subsystems $\{\mathcal{A}_i\}_{i=1}^{\infty}$ such that

$$\forall_{i,j} \partial\mathcal{P}_1 \cap \mathcal{A}_i = \partial\mathcal{P}_1 \cap \mathcal{A}_j \quad \text{and} \quad \lim_{i \rightarrow \infty} v(\mathcal{A}_i) = 0.$$

Then

$$\forall_{\mathcal{A}_i} Q_v(\mathcal{P}_1, \mathcal{A}_i) = Q(\mathcal{P}_1, \mathcal{A}_i) - Q_s(\mathcal{P}_1, \mathcal{A}_i)$$

and

$$\lim_{i \rightarrow \infty} Q_v(\mathcal{P}_1, \mathcal{A}_i) = \lim_{i \rightarrow \infty} [Q(\mathcal{P}_1, \mathcal{A}_i) - Q_s(\mathcal{P}_1, \mathcal{A}_i)] = 0,$$

according to the definition (3.13).

Hence

$$\lim_{i \rightarrow \infty} v(\mathcal{A}_i) = 0 \Rightarrow \lim_{i \rightarrow \infty} Q_v(\mathcal{P}_1, \mathcal{A}_i) = 0.$$

It means

$$\exists_{\tilde{\alpha} > 0} \forall_{(\mathcal{P}_1, \mathcal{P}_2) \in \text{sep} \mathbb{B} \times \mathbb{B}} |Q_v(\mathcal{P}_1, \mathcal{P}_2)| \leq \tilde{\alpha} v(\mathcal{P}_1) v(\mathcal{P}_2).$$

It remains to prove that $\tilde{\alpha} \geq \alpha$. With respect to the assumption that α, β are the least Lipschitz constants for $Q(\cdot, \cdot)$, we have

$$\exists_{(\mathcal{P}_1, \mathcal{P}_2) \in \text{sep} \mathbb{B} \times \mathbb{B}} \forall_{\epsilon > 0} |Q(\mathcal{P}_1, \mathcal{P}_2)| + \epsilon > \alpha v(\mathcal{P}_1) v(\mathcal{P}_2) + \beta s(\partial \mathcal{P}_1 \cap \mathcal{P}_2).$$

Simultaneously

$$|Q(\mathcal{P}_1, \mathcal{P}_2)| \leq |Q_s(\mathcal{P}_1, \mathcal{P}_2)| + |Q_v(\mathcal{P}_1, \mathcal{P}_2)| \leq \beta s(\partial \mathcal{P}_1 \cap \mathcal{P}_2) + \tilde{\alpha} v(\mathcal{P}_1) v(\mathcal{P}_2).$$

Hence

$$\forall_{\epsilon > 0} \alpha v(\mathcal{P}_1) v(\mathcal{P}_2) + \beta s(\partial \mathcal{P}_1 \cap \mathcal{P}_2) < \tilde{\alpha} v(\mathcal{P}_1) v(\mathcal{P}_2) + \beta s(\partial \mathcal{P}_1 \cap \mathcal{P}_2) + \epsilon,$$

and finally

$$\alpha \leq \tilde{\alpha}_{\blacksquare}$$

Now, let us consider the second possibility of the decomposition under the assumption on the existence of both limits (3.2)

$$(3.18) \quad \bar{Q}_v(\mathcal{P}_1, \mathcal{P}_2) := \lim_{i \rightarrow \infty} Q(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_i}).$$

For any two subsystems $\mathcal{P}_2, \mathcal{P}_3$ such that

$$v(\mathcal{P}_2 \Delta \mathcal{P}_3) \quad \text{and} \quad \mathcal{P}_1 \wedge \mathcal{P}_2 = \mathcal{P}_1 \wedge \mathcal{P}_3 = \emptyset,$$

we have

LEMMA

$$(3.19) \quad \bar{Q}_v(\mathcal{P}_1, \mathcal{P}_2) = \bar{Q}_v(\mathcal{P}_1, \mathcal{P}_3).$$

Proof. It is easy to see that we can always choose the family $\{\mathcal{A}_i\}_{i=1}^{\infty}$ in such a way that

$$v(\overline{\mathcal{P}_2 - \mathcal{A}_i} \Delta \overline{\mathcal{P}_3 - \mathcal{A}_i}) = 0.$$

For such a family, the lemma follows from the definition (3.18) \blacksquare

It means that the definition (3.18) yields the independence of the volume flux $\bar{Q}_v(\cdot, \cdot)$ on the surface interaction of \mathcal{P}_1 and \mathcal{P}_2 , i.e. the change of the surface $\partial \mathcal{P}_1 \cap \mathcal{P}_2$ does not have any bearing on $\bar{Q}_v(\cdot, \cdot)$ if it yields only the change of the surface measure of the contact surface.

Making use of the formula (3.11), we see that in general

$$(3.20) \quad \exists_{\mathcal{P}_2 \in \mathbb{B}} \bar{Q}_s(\mathcal{P}_1, \mathcal{P}_2) := Q(\mathcal{P}_1, \mathcal{P}_2) - \bar{Q}_v(\mathcal{P}_1, \mathcal{P}_2) \neq \lim_{i \rightarrow \infty} \bar{Q}_s(\mathcal{P}_1, \mathcal{A}_i).$$

The considered decomposition satisfies the continuity condition described by the following:

THEOREM

$$(3.21) \quad \forall_{(\mathcal{P}_1, \mathcal{P}_2) \in \text{sep} \mathbb{B} \times \mathbb{B}} |\bar{Q}_v(\mathcal{P}_1, \mathcal{P}_2)| \leq \alpha v(\mathcal{P}_1) v(\mathcal{P}_2),$$

$$\exists_{\tilde{\beta} > \beta} \forall_{(\mathcal{P}_1, \mathcal{P}_2) \in \text{sep} \mathbb{B} \times \mathbb{B}} |\bar{Q}_s(\mathcal{P}_1, \mathcal{P}_2)| \leq \tilde{\beta} s(\partial \mathcal{P}_1 \cap \mathcal{P}_2),$$

where α, β are again the least Lipschitz constant for $Q(\cdot, \cdot)$ and $\tilde{\beta}$ is the least Lipschitz constant for $\bar{Q}_s(\cdot, \cdot)$.

PROOF. The former inequality follows at once from the definition and the relation (3.10).

The latter requires an assumption on the completeness of the Lebesgue surface measure. Then the proof is similar to that of Theorem (3.17) ■

The second case in which we can define the surface and volume fluxes is based on the assumption of the existence of the limit

$$(3.22) \quad Q_s(\mathcal{P}_1, \mathcal{P}_2) := \lim_{i \rightarrow \infty} Q(\mathcal{P}_1, \mathcal{A}_i).$$

Investigating the first possibility considered above, we see that in our proofs we have not made use of the assumption on the existence of the limit of $\{Q(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_i})\}$. Hence the definition (3.22) yields the same consequences and, most of all, the lemma (3.15) and Theorem (3.17) hold.

On the other hand, the third case in which we assume the existence of the limit

$$(3.23) \quad \bar{Q}_s(\mathcal{P}_1, \mathcal{P}_2) := \lim_{i \rightarrow \infty} Q(\mathcal{P}_1, \overline{\mathcal{P}_2 - \mathcal{A}_i})$$

leads to the same consequences as the second possibility previously considered and, in particular, the lemma (3.19) and Theorem (3.21) hold.

Summing up the above considerations, we can say that the physically reasonable decomposition of the non-additive flux into the surface and volume parts is possible if the limit

$$\lim_{i \rightarrow \infty} Q(\mathcal{P}_1, \mathcal{A}_i)$$

exists for any family $\{\mathcal{A}_i\}_{i=1}^{\infty}$ of subsystems descending to the material surface. In such a case we arrive at the desirable relation for the surface part

$$(3.15) \quad \partial \mathcal{P}_1 \cap \mathcal{P}_2 = \partial \mathcal{P}_1 \cap \mathcal{P}_3 \Rightarrow Q_s(\mathcal{P}_1, \mathcal{P}_2) = Q_s(\mathcal{P}_1, \mathcal{P}_3),$$

that is, the surface part of the flux depends on the contact surface $\partial \mathcal{P}_1 \cap \mathcal{P}_2$ but not on the whole subsystem \mathcal{P}_2 . For this reason we deal further only with the decomposition, defined by the relation (3.13), assuming the existence of the limit on the right-hand side of this relation.

4. Representation of the surface flux

The classical theorems on the surface fluxes, such as the Cauchy's theorem on the linear relation between the unit normal vector of the surface and the density of the surface flux, require some additional assumptions. First of all, we make an assumption leading to the independence of the flux $Q_s(\cdot, \cdot)$ on the subsystems, being in the surface contact. As we have mentioned at the beginning of this paper, all results in axiomatic thermodynamics of continua are based on the biadditivity of the flux $Q(\cdot, \cdot)$. Our previous considerations have not required this property. Further, we assume the biadditivity of the surface part $Q_s(\cdot, \cdot)$. It is still far weaker than the biadditivity of $Q(\cdot, \cdot)$, but it turns out to be sufficient for many classical properties of $Q_s(\cdot, \cdot)$ to take place. Hence

AXIOM

$$(4.1) \quad \forall_{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \in \mathbf{B}} \mathcal{P}_1 \wedge \mathcal{P}_2 = \phi \wedge \mathcal{P}_1 \wedge \mathcal{P}_3 = \phi \wedge \mathcal{P}_3 \wedge \mathcal{P}_2 = \phi \Rightarrow \\ \Rightarrow Q_s(\mathcal{P}_1 \vee \mathcal{P}_2, \mathcal{P}_3) = Q_s(\mathcal{P}_1, \mathcal{P}_3) + Q_s(\mathcal{P}_2, \mathcal{P}_3) \wedge Q_s(\mathcal{P}_1, \mathcal{P}_2 \vee \mathcal{P}_3) = Q_s(\mathcal{P}_1, \mathcal{P}_2) + \\ + Q_s(\mathcal{P}_1, \mathcal{P}_3) \blacksquare$$

For any subsystem \mathcal{P}_3 such that

$$v(\mathcal{P}_3) = 0, \quad \mathcal{P}_3 < \mathcal{P}_1 \quad \text{and} \quad \partial \mathcal{P}_2 \cap \mathcal{P}_1 = \partial \mathcal{P}_2 \cap \mathcal{P}_3,$$

where \mathcal{P}_1 and \mathcal{P}_2 are any two non-overlapping subsystems, we have from Axiom (4.1)

$$Q_s(\mathcal{P}_1, \mathcal{P}_2) = Q_s(\overline{\mathcal{P}_1 - \mathcal{P}_3} \vee \mathcal{P}_3, \mathcal{P}_2) = Q_s(\overline{\mathcal{P}_1 - \mathcal{P}_3}, \mathcal{P}_2) + Q_s(\mathcal{P}_3, \mathcal{P}_2).$$

On the other hand, $\overline{\mathcal{P}_1 - \mathcal{P}_3} \cap \partial \mathcal{P}_2 = \phi$. Hence

$$Q_s(\mathcal{P}_1, \mathcal{P}_3) = Q_s(\mathcal{P}_3, \mathcal{P}_2).$$

This proves the following:

LEMMA. For any descending family of subsystems $\{\mathcal{B}_j\}_{j=1}^\infty$ satisfying the conditions

$$(4.2) \quad \begin{aligned} & \text{i) } \quad \forall_j \mathcal{B}_j < \mathcal{P}_1 \quad \text{and} \quad \mathcal{B}_j \cap \partial \mathcal{P}_2 = \mathcal{P}_1 \cap \partial \mathcal{P}_2, \\ & \text{ii) } \quad \bigcap_{j=1}^\infty \mathcal{B}_j = \overline{\text{int } \partial \mathcal{P}_2 \cap \mathcal{P}_1}, \end{aligned}$$

we have

$$\lim_{j \rightarrow \infty} Q_s(\mathcal{B}_j, \mathcal{P}_2) = Q_s(\mathcal{P}_1, \mathcal{P}_2) \blacksquare$$

Taking into account the definition (3.13) and the above Lemma, we see that there exists a function $H(\cdot)$, defined on the collection of material surfaces, induced by the family \mathbf{B} , such that

$$(4.3) \quad \forall_{(\mathcal{P}_1, \mathcal{P}_2) \in \text{sepB} \times \mathbf{B}} H(\partial \mathcal{P}_1 \cap \mathcal{P}_2) = Q_s(\mathcal{P}_1, \mathcal{P}_2) \quad \text{and} \quad |H(\partial \mathcal{P}_1 \cap \mathcal{P}_2)| \leq \beta s(\partial \mathcal{P}_1 \cap \mathcal{P}_2),$$

with the property

$$(4.4) \quad H(s_1 \cup s_2) = H(s_1) + H(s_2),$$

where s_1 and s_2 are any two non-overlapping material surfaces.

However, without further assumptions the function $H(\cdot)$ is not balanced, i.e. the change of the orientation of the surface $\partial \mathcal{P}_1 \cap \mathcal{P}_2$ yields in general the different value of $H(\cdot)$ in contrast to the classical surface fluxes. This means that

$$\exists_{(\mathcal{P}_1, \mathcal{P}_2) \in \text{sepB} \times \mathbf{B}} H(\partial \mathcal{P}_1 \cap \mathcal{P}_2) + H(\partial \mathcal{P}_2 \cap \mathcal{P}_1) \neq 0.$$

The balance relation for $H(\cdot)$ can be proved if we assume in addition the Lipschitz volume continuity of $\dot{E}(\cdot)$. Namely,

THEOREM. (Noll). *If the derivative of the state function is Lipschitz volume continuous, i.e.*

$$(4.5) \quad \exists_{\gamma > 0} \forall_{\mathcal{P} \in \mathbf{B}} |\dot{E}(\mathcal{P})| \leq \gamma v(\mathcal{P}),$$

then the surface flux $Q_v(\cdot, \cdot)$ is balanced:

$$\forall_{(\mathcal{P}_1, \mathcal{P}_2) \in \text{sepB} \times \mathbf{B}} H(\partial \mathcal{P}_1 \cap \mathcal{P}_2) = -H(\partial \mathcal{P}_2 \cap \mathcal{P}_1).$$

P r o o f. Let us consider the function describing long-range interactions between two subsystems $\mathcal{A}_i, \mathcal{B}_j$, where \mathcal{A}_i is the member of the family defined by Eq. (3.1), while the subsystem \mathcal{B}_j is the member of the family defined in Lemma (4.2). Then

$$\dot{E}(\mathcal{A}_i \vee \mathcal{B}_j) - \dot{E}(\mathcal{A}_i) - \dot{E}(\mathcal{B}_j) = Q(\mathcal{A}_i \vee \mathcal{B}_j, \mathcal{A}_i^e \wedge \mathcal{B}_j^e) - Q(\mathcal{A}_i, \mathcal{A}_i^e) - Q(\mathcal{B}_j, \mathcal{B}_j^e).$$

According to the relation (3.2) and Axiom (4.1), we have

$$\begin{aligned} \dot{E}(\mathcal{A}_i \vee \mathcal{B}_j) - \dot{E}(\mathcal{A}_i) - \dot{E}(\mathcal{B}_j) = & -H(\partial\mathcal{A}_i \cap \mathcal{B}_j) - H(\partial\mathcal{B}_j \cap \mathcal{A}_i) \\ & + Q_v(\mathcal{A}_i \vee \mathcal{B}_j, \mathcal{A}_i^e \wedge \mathcal{B}_j^e) - Q_v(\mathcal{A}_i, \mathcal{A}_i^e) - Q_v(\mathcal{B}_j, \mathcal{B}_j^e). \end{aligned}$$

Assuming the existence of the pertinent limits and making use of the continuity assumptions, we obtain at once

$$H(\partial\mathcal{P}_1 \cap \mathcal{P}_2) + H(\partial\mathcal{P}_2 \cap \mathcal{P}_1) = 0 \blacksquare$$

A similar theorem has been proved by W. Noll for biadditive fluxes and additive state functions.

We can summarize the above considerations as follows: when all above assumptions hold, the flux $Q(\cdot, \cdot)$ has the form

$$(4.6) \quad \begin{aligned} \forall_{(\mathcal{P}_1, \mathcal{P}_2) \in \text{sepB} \times \text{B}} \quad Q(\mathcal{P}_1, \mathcal{P}_2) &= H(\partial\mathcal{P}_1 \cap \mathcal{P}_2) + Q_v(\mathcal{P}_1, \mathcal{P}_2); \\ H(\partial\mathcal{P}_1 \cap \mathcal{P}_2) &= -H(\partial\mathcal{P}_2 \cap \mathcal{P}_1); \\ \exists_{\beta > 0} \quad |H(\partial\mathcal{P}_1 \cap \mathcal{P}_2)| &\leq \beta s(\partial\mathcal{P}_1 \cap \mathcal{P}_2); \\ \forall_{\substack{\mathcal{J}_1, \mathcal{J}_2 - \text{mat. surf.} \\ \mathcal{J}_1 \cap \mathcal{J}_2 = \emptyset}} \quad H(\mathcal{J}_1 \cap \mathcal{J}_2) &= H(\mathcal{J}_1) + H(\mathcal{J}_2); \\ \exists_{\delta > 0} \quad |Q_v(\mathcal{P}_1, \mathcal{P}_2)| &\leq \delta v(\mathcal{P}_1)v(\mathcal{P}_2). \end{aligned}$$

The above decomposition admits long-range interactions, many body actions being described solely by $Q_v(\cdot, \cdot)$.

The biadditivity assumption on $Q_s(\cdot, \cdot)$ yields some further properties of the surface flux $H(\cdot)$. Considerations similar to these of M. E. GÜRTIN and W. O. WILLIAMS [1967] lead to the localization of $H(\cdot)$ on the material surface. According to Theorem 7 of that paper (see also W. O. WILLIAMS [1970], Proposition 6), we have

$$(4.7) \quad \forall_{\substack{\mathbf{n}, |\mathbf{n}|=1}} \exists q(\cdot, \mathbf{n}) : \mathcal{J} \rightarrow \mathbb{R} \quad \text{such that} \quad H(\mathcal{J}) = \int_{\mathcal{J}} q(x, \mathbf{n}) ds_x,$$

where \mathcal{J} is a material surface and $\mathbf{n}(x)$ denotes the unit normal to \mathcal{J} at $x \in \mathcal{J}$.

Now, the balance equation can be written in the following form:

$$(4.8) \quad \forall_{\mathcal{P} \in \text{B}} \int_{\partial\mathcal{P}} q(x, \mathbf{n}) ds_x = \dot{E}(\mathcal{P}) - Q_v(\mathcal{P}, \mathcal{P}^e).$$

Making use of the continuity assumptions of Theorem (4.5), we see at once that the limit

$$(4.9) \quad \lim_{i \rightarrow \infty} \frac{1}{v(\mathcal{P}_i)} \{ \dot{E}(\mathcal{P}_i) - Q_v(\mathcal{P}_i, \mathcal{P}_i^e) \} = \lim_{i \rightarrow \infty} \frac{1}{v(\mathcal{P}_i)} \int_{\partial\mathcal{P}_i} q(x, \mathbf{n}) ds_x$$

exists for any Vitali sequence $\{\mathcal{P}_i\}_{i=1}^{\infty}$ of subsystems converging to the point x . The limit on the right-hand side is called by W. O. WILLIAMS [1967] a divergence

$$(4.10) \quad \operatorname{div} \mathbf{q}(x) := \lim_{i \rightarrow \infty} \frac{1}{v(\mathcal{P}_i)} \int_{\partial \mathcal{P}_i} \mathbf{q}(x, \mathbf{n}) ds_x.$$

Hence the balance equation takes the form

$$(4.11) \quad \forall_{\mathcal{P} \in \mathcal{B}} \dot{E}(\mathcal{P}) = \int_{\mathcal{P}} \operatorname{div} \mathbf{q}(x) dv_x + Q_v(\mathcal{P}, \mathcal{P}^e).$$

In the general case, this equation cannot be localized with respect to the nonlocalities described by $Q_v(\cdot, \cdot)$ and $\dot{E}(\cdot)$. The biadditivity of $Q_v(\cdot, \cdot)$ leads to certain non-unique localizations (W. BARAŃSKI [1972, 1974]), delivered by M. E. GURTIN and W. O. WILLIAMS.

References

1. W. NOLL, *The foundations of classical mechanics in the light of recent advances in continuum mechanics*, in: *The axiomatic method with special reference to geometry and physics*, 1959.
2. M. E. GURTIN, W. O. WILLIAMS, *An axiomatic foundation for continuum thermodynamics*, Arch. Rat. Mech. Anal., **26**, 2, 1967.
3. W. O. WILLIAMS, *Thermodynamics of rigid continua*, Arch. Rat. Mech. Anal., **36**, 4, 1970.
4. W. O. WILLIAMS, *Axioms for work and energy in general continua, I. Smooth velocity fields*, Arch. Rat. Mech. Anal., **42**, 2, 1971.
5. W. BARAŃSKI, *Additivity of mechanical power and the principle of stress*, Arch. Mech., **24**, 4, 1972.
6. W. BARAŃSKI, *A continuous model of the material universum*, Bull. Acad. Polon. Sci., Série Sci. Tech., **22**, 10, 1974.
7. K. WILMAŃSKI, *Podstawy termodynamiki fenomenologicznej*, PWN, Warszawa 1974.
8. D. G. B. EDELEN, *Nonlocal field theories*, in: *Continuum physics*, vol. IV, Academic Press Inc., 1976.
9. K. WILMAŃSKI, *On the continuity of fluxes in axiomatic thermodynamics*, Letters in App. Engng. Sci., **16**, 1978.

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