

Modulational instability of nonlinear surface waves in magnetohydrodynamics

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THE AVERAGED Lagrangian method is used to investigate the magnetohydrodynamic effects on the evolution of a finite-amplitude gravity wave on the surface of a conducting liquid of large depth. The magnetohydrodynamic effects are found to inhibit the breakdown of a finite-amplitude uniform wavetrain into a series of wavegroups.

STUDIES of surface waves in the sphere of magnetohydrodynamics have been made by SAVAGE [3, 4], DEBNATH [1] and SHIVAMOGGI [5, 6]. The propagation of plane waves in ferro-fluids in the presence of a tangential magnetic field was investigated experimentally by ZELAZO and MELCHER [10], who showed that the magnetic field has a stabilizing influence on the waves at the fluid interface. The stabilizing effect of the tangential magnetic field on the nonlinear propagation of a wave packet on a fluid surface in magnetohydrodynamics was theoretically shown by MALIK and SINGH [2] who used the method of multiple scales. However, the latter approach becomes somewhat unwieldy; MALIK and SINGH [2] actually resorted to numerical work to deduce the final result. The purpose of this note is to show that the averaged Lagrangian method (WHITHAM: [7, 8]; YUEN and LAKE: [9]) affords a more elegant approach to this problem. Indeed, in this approach there is no need to resort to numerical work to show the stabilizing effect of the magnetic field on the nonlinear propagation of a wave packet on the fluid surface.

Consider an initially-quiet, infinitely-conducting liquid subjected to a gravitational field g and confined in the region $y < 0$ by a vacuum magnetic field B_0 aligned with the surface of the liquid (Fig. 1). A variational principle for the motion of the interface between

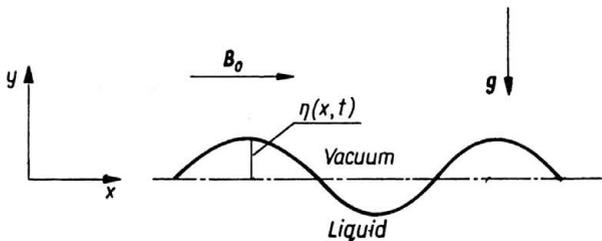


FIG. 1.

an infinitely-conducting liquid and a vacuum magnetic field was given by SHIVAMOGGI [6], which is

$$(1) \quad \delta J = \delta \int_{t_1}^{t_2} \int_{x_1}^{x_2} L dx dt = 0,$$

where,

$$(2) \quad L = \int_{-\infty}^{\eta(x,t)} \left[\phi_t + \frac{1}{2} (\phi_x^2 + \phi_y^2) + gy \right] dy - \frac{1}{\rho} \int_{\eta(x,t)}^{\infty} \frac{1}{2} (\Psi_x^2 + \Psi_y^2) dy,$$

Ψ being the total magnetic field potential; ϕ being the velocity potential, ρ being the density of the liquid. Here $\phi(x, y, t)$, $\Psi(x, y, t)$ and $\eta(x, t)$ are allowed to vary, subject to the restrictions $\delta\Phi$, $\delta\Psi$ and $\delta\eta = 0$ at $x = x_1, x_2$ and $t = t_1, t_2$.

Following the usual procedure of calculus of variations, Eqs. (1) and (2) give

$$(3) \quad y < \eta(x, t): \quad \phi_{xx} + \phi_{yy} = 0,$$

$$(4) \quad y > \eta(x, t): \quad \Psi_{xx} + \Psi_{yy} = 0,$$

$$(5) \quad y = \eta(x, t): \quad -\eta_t + \phi_y - \eta_x \phi_x = 0,$$

$$(6) \quad y = \eta(x, t): \quad \Psi_y = \eta_x \Psi_x,$$

$$(7) \quad y \Rightarrow \eta(x, t): \quad \phi_t + \frac{1}{2} (\phi_x^2 + \phi_y^2) + gy + \frac{1}{2\rho} (\Psi_x^2 + \Psi_y^2) = 0,$$

$$(8) \quad y \Rightarrow -\infty: \phi_y \Rightarrow 0,$$

$$(9) \quad y \Rightarrow \infty: \Psi_y \Rightarrow 0.$$

Equation (5) described the kinematic condition on the velocity field. Equation (6) describes the constraint of frozen-in lines of force in the liquid on the vacuum magnetic field (so that the interface remains a magnetic field line even in a perturbed state). Equation (7) describes the dynamic condition of force balance at the interface.

For the present problem we have

$$(10) \quad \Psi = B_0(x + \psi).$$

Let us now consider a finite-amplitude stationary Alfvén-gravity wave of frequency ω_0 and wavenumber k_0 propagating in the X -direction and superpose on it a slowly-varying weak modulation and study the evolution of such a modulation.

If, following WHITHAM (1970), we assume that the wave can still be taken to be sinusoidal locally, i.e.

$$(11) \quad \eta = a \cos \theta$$

but with amplitude and phase varying slowly in x and t , i.e.

$$(12) \quad \begin{aligned} a &= a(x, t), \\ \theta &= \theta(x, t) = k_0 x - \omega_0 t + \varphi(x, t), \end{aligned}$$

then we may introduce a generalized frequency ω and wave-number k

$$(13) \quad \begin{aligned} \omega &= -\theta_t = \omega_0 - \varphi_t, \\ k &= \theta_x = k_0 + \varphi_x. \end{aligned}$$

We have from the relations (13) a compatibility condition

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0$$

Using the relations (3)–(6), (8)–(10), we obtain

$$(15) \quad \eta = a \cos \theta + ka^2 \cos 2\theta,$$

$$(16) \quad \phi = \left[\frac{\omega a}{k} \sin \theta + \frac{\omega a_x}{k^2} (1 - ky) \cos \theta + \frac{a_t}{k} \cos \theta \right] e^{ky} + \left(\frac{\omega a^2}{2} \sin 2\theta \right) e^{2ky},$$

$$(17) \quad \psi = \left[a \sin \theta + \frac{a_x}{k} (1 + ky) \cos \theta \right] e^{-ky} + \left(\frac{3}{2} ka^2 \sin 2\theta \right) e^{-2ky}.$$

In order to obtain the equations describing the long-time evolution of the wave, we use Eqs. (15)–(17) in Eq. (2), and calculate the averaged Lagrangian L ,

$$(18) \quad \mathcal{L} = \frac{1}{2\pi} \int_0^{2\pi} L d\theta = \left(-\frac{\omega^2}{4k} + \frac{g}{4} + \frac{kV_A^2}{4} \right) a^2 + \frac{a_t^2}{4k} + \frac{aa_{tt}}{2k} + \frac{\omega a_x a_t}{4k^2} \\ + \frac{3}{8} \left(\frac{\omega^2}{k^3} + \frac{V_A^2}{3k} \right) aa_{xx} + \frac{3}{4} \frac{\omega a a_{xt}}{k^2} + \left(\frac{\omega^2}{8k^3} - \frac{V_A^2}{8k} \right) a_x^2 + \left(\frac{\omega^2}{8} k - \frac{1}{8} k^3 V_A^2 \right) a^4,$$

where

$$V_A^2 \equiv \frac{B_0^2}{\rho}.$$

Then the variation of \mathcal{L} with respect to θ gives

$$(19) \quad \delta\theta: \quad \frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \omega} \right) - \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial k} \right) = 0$$

which, on using Eq. (14) gives in turn

$$(20) \quad \frac{\partial}{\partial t} (a^2) + \frac{\partial}{\partial x} \left(\frac{d\omega}{dk} a^2 \right) = 0.$$

Next, the variation of \mathcal{L} with respect to a gives

$$(21) \quad \delta a: \quad \frac{\partial \mathcal{L}}{\partial a} = 0$$

which, on using Eq. (20), gives in turn

$$(22) \quad \omega = \omega_0 \left[1 + \frac{k^2}{2} \left(1 - \frac{k^2 V_A^2}{\omega_0^2} \right) a^2 - \frac{1}{2\omega_0} \frac{d^2 \omega_0}{dk^2} \left\{ 1 + \frac{2\omega_0 V_A^2}{g^2} \right\} \left(\frac{a_{xx}}{a} \right) \right],$$

where

$$(23) \quad \omega_0^2 = gk + k^2 V_A^2.$$

For weak modulations, using Eq. (22), we may write

$$(24) \quad \omega = \omega_0 + \frac{\partial \omega_0}{\partial k_0} (k - k_0) + \frac{1}{2} \frac{\partial^2 \omega_0}{\partial k_0^2} \left[(k - k_0)^2 - (1 + \beta) \left(\frac{a_{xx}}{a} \right) \right] + \frac{\partial \omega}{\partial a_0^2} (a^2 - a_0^2)$$

where

$$\beta \equiv \frac{2\omega_0^2 V_A^2}{g^2}.$$

Using the relations (13), we obtain from Eqs. (22) and (20)

$$(25) \quad \varphi_t + \frac{\partial \omega_0}{\partial k_0} \varphi_x + \frac{1}{2} \frac{\partial^2 \omega_0}{\partial k_0^2} \left[\varphi_x^2 - (1 + \beta) \frac{a_{xx}}{a} \right] + \frac{\omega_0 k_0^2}{2} (1 - \alpha)(a^2 - a_0^2) = 0,$$

$$(26) \quad (a^2)_t + \frac{\partial \omega_0}{\partial k_0} (a^2)_x + \frac{\partial^2 \omega_0}{\partial k_0^2} (\varphi_x a^2)_x = 0,$$

where

$$(27) \quad \alpha \equiv \frac{k^2 V_A^2}{\omega_0^2}.$$

Putting

$$(28) \quad \chi = a e^{i\varphi}.$$

We obtain from Eqs. (25) and (26) a modified nonlinear Schrödinger equation

$$(29) \quad i \left(\frac{\partial \chi}{\partial t} + \frac{\partial \omega_0}{\partial k_0} \frac{\partial \chi}{\partial x} \right) + \frac{1}{2} \frac{\partial \omega_0}{\partial k_0^2} \frac{\partial^2}{\partial x^2} (\chi + \beta |\chi|) = (1 - \alpha) |\chi|^2 \chi$$

describing the modulation of the magnetohydrodynamic surface wave. In order to investigate the stability of this modulation, let us put

$$(30) \quad \chi = \sqrt{\varrho(\xi, t)} e^{i\sigma(\xi, t)},$$

where

$$(31) \quad \xi \equiv x - \frac{\partial \omega_0}{\partial k_0} t.$$

Then we obtain from Eq. (29)

$$(32) \quad \frac{\partial \varrho}{\partial t} + \frac{\partial^2 \omega_0}{\partial k_0^2} \frac{\partial}{\partial \xi} \left(\varrho \frac{\partial \sigma}{\partial \xi} \right) = 0,$$

$$(33) \quad -\omega_0(1 - \alpha)(\varrho - \varrho_0) - \frac{\partial \sigma}{\partial t} + \frac{1}{4\varrho} \frac{\partial^2 \omega_0}{\partial k_0^2} \left[(1 + \beta) \left\{ \frac{\partial^2 \varrho}{\partial \xi^2} - \frac{1}{2\varrho} \left(\frac{\partial \varrho}{\partial \xi} \right)^2 \right\} - 2\varrho \left(\frac{\partial \sigma}{\partial \xi} \right)^2 \right] = 0.$$

Let us put further

$$(34) \quad \begin{aligned} \varrho &= \varrho_0 + \varrho_1(\xi, t), \\ \sigma &= \sigma_1(\xi, t) \end{aligned}$$

and assume

$$(35) \quad \varrho_1(\xi, t), \sigma_1(\xi, t) \sim e^{i(K\xi - \Omega t)}.$$

Linearizing in ϱ_1 and σ_1 , we obtain from Eqs. (32) and (33)

$$(36) \quad i\Omega \varrho_1 + \frac{\partial^2 \omega_0}{\partial k_0^2} \varrho_0 K^2 \sigma_1 = 0,$$

$$(37) \quad - \left[\omega_0(1 - \alpha) + \frac{K^2}{4\varrho_0} \frac{\partial^2 \omega_0}{\partial k_0^2} (1 + \beta) \right] \varrho_1 + i\Omega \sigma_1 = 0.$$

from which

$$(38) \quad \Omega^2 = \omega_0 K^2 \frac{\partial^2 \omega_0}{\partial k_0^2} \left[\frac{K^2}{4\omega_0} \frac{\partial^2 \omega_0}{\partial k_0^2} (1 + \beta) + (1 - \alpha) |\chi_0|^2 \right].$$

Noting from Eq. (23) that

$$(39) \quad \frac{\partial^2 \omega_0}{\partial k_0^2} = - \frac{g^2}{4\omega_0^3} < 0$$

and using Eqs. (24) and (27), we observe in Eq. (38) that the threshold value of the wave amplitude $|\chi_0|$ to cause the instability increases when $V_A \neq 0$. This clearly implies the stabilizing effect of the magnetic field on the gravity waves.

References

1. L. DEBNATH, *Plasma Phys.*, **19**, 263, 1977.
2. S. K. MALIK, M. SINGH, *Q. Appl. Math.*, **43**, 57, 1985.
3. M. D. SAVAGE, *J. Plasma Phys.*, **1**, 229, 1967.
4. M. D. SAVAGE, *J. Fluid Mech.*, **42**, 289, 1970.
5. B. K. SHIVAMOGGI, *J. Plasma Phys.*, **27**, 321, 1982.
6. B. K. SHIVAMOGGI, *Q. Appl. Math.*, **41**, 31, 1983.
7. G. B. WHITHAM, *Proc. Roy. Soc.*, **A229**, 6, London 1967.
8. G. B. WHITHAM, *J. Fluid Mech.*, **44**, 373, 1970.
9. H. C. YUEN, B. M. LAKE, *Phys. Fluids*, **18**, 956, 1975.
10. R. E. ZELAZO, J. R. MELCHER, *J. Fluid Mech.*, **39**, 1, 1969.

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